21-640: Functional Analysis

Spring 2023 • Prof. Giovanni Leoni

Notes by Fan Pu Zeng

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21-640: Functional Analysis Spring 2023

Lecture 1 — January 18, 2023

Prof. Giovanni Leoni *Scribe: Fan Pu Zeng*

1 Overview

Prerequisites.

- Measure theory (important for examples).
- Basic topology: topological spaces, open sets, closed sets, continuity, compact sets.

The course studies the following classes of spaces, each contained in the next.

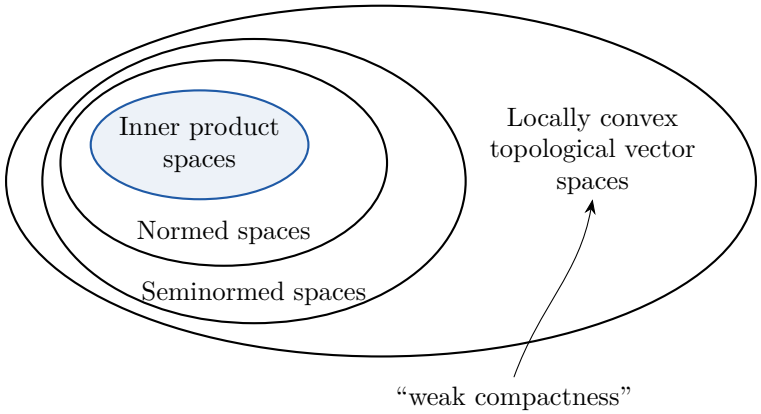

Figure 1: The classes of spaces in the course, from inner product spaces (innermost) to locally convex topological vector spaces (outermost).

We lose compactness as we go out from inner product spaces; for the outermost class the relevant notion is "weak compactness".

2 Inner Product Spaces

Definition (Inner product)**.** Let X be a vector space over $\mathbb{R}$. A function

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \langle x, y \rangle ,$$

is an *inner product* if

(1) $\langle x, x \rangle \geq 0$ for every $x \in X$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$;

(3) $\langle tx + sy, z \rangle = t \langle x, z \rangle + s \langle y, z \rangle$ for all $x, y, z \in X$ and $t, s \in \mathbb{R}$.

2.1 Examples

Example 1 (Euclidean space $\mathbb{R}^N$)**.** We write vectors in $\mathbb{R}^N$ in boldface, $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$. The inner product is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N.$$

Example 2 (L^2 spaces)**.** Let $E \subseteq \mathbb{R}^N$ be a measurable set with respect to Lebesgue measure, and let

$$X = \Big\{ f : E \to \mathbb{R} \ : \ f \text{ measurable and } \int_E |f(\mathbf{x})|^2 \, d\mathbf{x} < \infty \Big\}.$$

For $f, g \in X$, define

$$\langle f, g \rangle := \int_E f(\mathbf{x}) \, g(\mathbf{x}) \, d\mathbf{x}.$$

Well defined. Since $ab \le a^2 + b^2$,

$$\int_E |f(\mathbf{x}) \, g(\mathbf{x})| \, d\mathbf{x} \le \int_E |f(\mathbf{x})|^2 \, d\mathbf{x} + \int_E |g(\mathbf{x})|^2 \, d\mathbf{x} < \infty.$$

Is the L^2 space an inner product space?

(1) Non-negativity: $\langle f, f \rangle = \int_E |f(\mathbf{x})|^2 \, d\mathbf{x} \ge 0$. But if $\int_E |f(\mathbf{x})|^2 \, d\mathbf{x} = 0$, then we only know that there is $G \subseteq E$ with $\mathcal{L}^N(G) = 0$ (a set of Lebesgue measure 0) such that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in E \setminus G$. So f need not be the zero function.

(2) Symmetry: ok.

(3) Linearity: ok.

So property (1) fails, and this space is not an inner product space.

2.2 Fix: take quotients

Let $f, g \in X$. We say f is *equivalent* to g, written $f \sim g$, if there is $G \subseteq E$ with $\mathcal{L}^N(G) = 0$ such that

$$f(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \in E \setminus G.$$

Then $\sim$ is an equivalence relation (check).

Definition ($L^2(E)$)**.**

$$L^2(E) := X/\!\sim \; = \{[f] : f \in X\}, \qquad \text{where } [f] = \{g \in X : g \sim f\}.$$

Important: $[f]$ is *not* a function. It is an equivalence class of functions. "$f \in L^2(E)$" means $[f] \in L^2(E)$, but many times we drop the brackets. For example, with $N = 1$ (that is, on $\mathbb{R}$), $[f](3)$ makes no sense.

Definition. If $[f], [g] \in L^2(E)$, define

$$\langle [f], [g] \rangle := \int_E f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}.$$

Is it well defined? (Changing the representative should not change the value.) Yes, because Lebesgue integration does not see sets of measure zero.

If

$$\langle [f], [f] \rangle = \int_E \big(f(\mathbf{x})\big)^2\, d\mathbf{x} = 0,$$

then $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in E \setminus G$, where $\mathcal{L}^N(G) = 0$. So $f \sim 0$, and $[f] = [0]$. Hence this is an inner product.

2.3 Fractional Sobolev spaces

Let $0 < s < 1$ and let $U \subseteq \mathbb{R}^N$ be open.

Definition (Fractional Sobolev space)**.** The *fractional Sobolev space* $H^s(U)$ is the space of all "functions" $[f] \in L^2(U)$ such that

$$\int_U \int_U \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}}\, d\mathbf{x}\, d\mathbf{y} < \infty.$$

Intuition: this requires some regularity of the functions; continuity is not enough.

For $[f], [g] \in H^s(U)$,

$$\langle [f], [g] \rangle = \int_U \int_U \frac{\big(f(\mathbf{x}) - f(\mathbf{y})\big)\big(g(\mathbf{x}) - g(\mathbf{y})\big)}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}}\, d\mathbf{x}\, d\mathbf{y} + \int_U f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}$$

is an inner product.

2.4 Sobolev spaces

Let $U \subseteq \mathbb{R}^N$ be open, $[f] \in L^2(U)$ and $i = 1, \ldots, N$.

Definition (Weak derivative)**.** We say $[f]$ has a *weak i-th derivative* in $L^2(U)$ if there is $[g_i] \in L^2(U)$ such that

$$\int_U f(\mathbf{x}) \frac{\partial \varphi}{\partial x_i}(\mathbf{x})\, d\mathbf{x} = -\int_U g_i(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x}$$

for all $\varphi \in C_c^1(U)$ (continuously differentiable with compact support). This is a kind of integration by parts. Then $[g_i]$ is the *weak i-th partial derivative*, denoted $\dfrac{\partial [f]}{\partial x_i}$.

One can prove that the weak partial derivative is unique if it exists.

Definition (Sobolev space)**.** The *Sobolev space* $H^1(U)$ is the space of all "functions" $[f] \in L^2(U)$ for which $\dfrac{\partial [f]}{\partial x_i} \in L^2(U)$ exists for all $i = 1, \ldots, N$.

Its inner product is

$$\langle [f], [g] \rangle = \int_U f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x} + \sum_{i=1}^{N} \int_U f_i(\mathbf{x})\, g_i(\mathbf{x})\, d\mathbf{x},$$

where $f_i \in \dfrac{\partial [f]}{\partial x_i}$ and $g_i \in \dfrac{\partial [g]}{\partial x_i}$.

Lecture 2 — January 20, 2023

Prof. Giovanni Leoni *Scribe: Fan Pu Zeng*

1 Recall: Inner Products

Definition (Inner product)**.** Let X be a vector space over $\mathbb{R}$. A function

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \langle x, y \rangle ,$$

is an *inner product* if

(1) $\langle x, x \rangle \geq 0$ for every $x \in X$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$;

(3) $\langle tx + sy, z \rangle = t \langle x, z \rangle + s \langle y, z \rangle$ for all $x, y, z \in X$ and $t, s \in \mathbb{R}$.

2 The Cauchy–Schwarz Inequality

Theorem 1 (Cauchy–Schwarz inequality)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then*

$$| \langle x, y \rangle | \leq \|x\| \, \|y\| \qquad \text{for all } x, y \in X,$$

where $\|x\| := \sqrt{\langle x, x \rangle}$.

Example 3. Let $E \subseteq \mathbb{R}^N$ be Lebesgue measurable. Then

$$\left| \int_E f(\mathbf{x}) \, g(\mathbf{x}) \, d\mathbf{x} \right| \leq \left(\int_E |f(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_E |g(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}$$

for all $[f], [g] \in L^2(E)$. (Here the inner product is between equivalence classes of functions.)

Proof of Theorem 1. Let $t \in \mathbb{R}$ and $x, y \in X$. The numbers over each step refer to the properties of the inner product. Then

$$\begin{aligned} 0 &\overset{(1)}{\leq} \langle x + ty, x + ty \rangle \\ &\overset{(3)}{=} \langle x, x + ty \rangle + t \langle y, x + ty \rangle \\ &\overset{(2)}{=} \langle x + ty, x \rangle + t \langle x + ty, y \rangle \\ &\overset{(3)}{=} \langle x, x \rangle + t \langle y, x \rangle + t \langle x, y \rangle + t^2 \langle y, y \rangle \\ &= \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 , \end{aligned}$$

so

$$\boxed{\|x\|^2 + 2t\,\langle x, y\rangle + t^2\,\|y\|^2 \geq 0} \qquad \text{for all } t \in \mathbb{R}.$$

Assume $y \neq 0$, and take

$$t = -\frac{\langle x, y\rangle}{\|y\|^2}$$

(the denominator is non-zero). Then

$$\|x\|^2 - 2\frac{\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2}{\|y\|^2} \geq 0,$$

so

$$\|x\|^2\,\|y\|^2 \geq \langle x, y\rangle^2, \qquad \text{that is,} \qquad \|x\|\,\|y\| \geq |\,\langle x, y\rangle\,|.$$

If $y = 0$, both sides are 0 and we get $0 = 0$. ■

Remark 1. Taking $t = 1$ and $t = -1$ above gives

$$\|x + y\|^2 = \|x\|^2 + 2\,\langle x, y\rangle + \|y\|^2, \tag{1}$$

$$\|x - y\|^2 = \|x\|^2 - 2\,\langle x, y\rangle + \|y\|^2. \tag{2}$$

Summing (1) and (2) gives the *parallelogram law*:

$$\boxed{\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2.}$$

3 The Norm Induced by an Inner Product

Corollary 1. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then $\|x\| := \sqrt{\langle x, x\rangle}$ is a norm.*

Proof. To prove it is a norm, we need to show:

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(2) (homogeneity) $\|tx\| = |t|\,\|x\|$ for all $t \in \mathbb{R}$ and $x \in X$;

(3) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Proof of (1). $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$. Moreover, $\|x\| = \sqrt{\langle x, x\rangle} = 0$ if and only if $\langle x, x\rangle = 0$, if and only if $x = 0$.

Proof of (2). Using properties (2) and (3) of the inner product,

$$\|tx\| = \sqrt{\langle tx, tx\rangle} = \sqrt{t^2\,\langle x, x\rangle} = |t|\sqrt{\langle x, x\rangle} = |t|\,\|x\|.$$

Proof of (3). By (1) (the case $t = 1$) and the Cauchy–Schwarz inequality,

$$\|x + y\|^2 = \|x\|^2 + 2\,\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2,$$

so $\|x + y\|^2 \leq (\|x\| + \|y\|)^2$, and $\|x + y\| \leq \|x\| + \|y\|$. ■

Hence $d(x, y) = \|x - y\|$ is a metric.

4 Hilbert Spaces

Definition (Hilbert space)**.** $(X, \langle\cdot,\cdot\rangle)$ is a *Hilbert space* if (X, d) is a complete metric space. That is, every Cauchy sequence in X has a limit in X.

Definition (Cauchy sequence)**.** Let (X, d) be a metric space. A sequence $\{x_m\}_m$ is a *Cauchy sequence* if

$$\lim_{m,k\to\infty} d(x_m, x_k) = 0.$$

Example 4. The following are complete, and hence Hilbert spaces:

- $L^2(E)$;
- the Sobolev space $H^1(U)$ (on the homework);
- the fractional Sobolev space $H^s(U)$ (on the homework; play with the Cauchy–Schwarz inequality).

5 Projection onto Closed Convex Sets

Theorem 2. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $G \subseteq X$ be a closed, nonempty, convex set. Then for every $x \in X$ there is a* unique *$y \in G$ such that*

$$\|x - y\| = \inf_{z\in G} \|x - z\| = \operatorname{dist}(x, G).$$

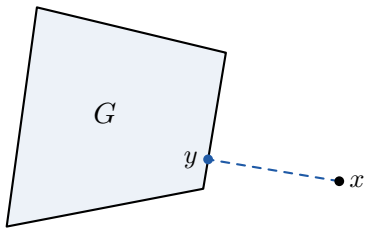

Figure 2: The point $y \in G$ closest to x.

Proof. Let

$$d = \inf_{z\in G} \|x - z\|.$$

Find $z_m \in G$ such that $\lim_{m\to\infty} \|x - z_m\| = d$, for instance with $d \le \|x - z_m\| < d + \frac{1}{m}$.

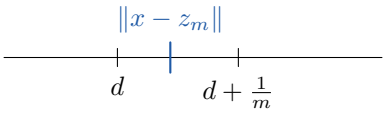

Figure 3: The distance $\|x - z_m\|$ lies between d and $d + \frac{1}{m}$.

Claim 1. $\{z_m\}$ is a Cauchy sequence.

Apply the parallelogram law to $\frac{x-z_m}{2}$ and $\frac{x-z_k}{2}$:

$$\left\|\frac{x-z_m}{2}+\frac{x-z_k}{2}\right\|^2+\left\|\frac{x-z_m}{2}-\frac{x-z_k}{2}\right\|^2=2\left\|\frac{x-z_m}{2}\right\|^2+2\left\|\frac{x-z_k}{2}\right\|^2,$$

which is equivalent to

$$\left\|x-\frac{z_m+z_k}{2}\right\|^2+\frac{1}{4}\|z_m-z_k\|^2=\frac{1}{2}\|x-z_m\|^2+\frac{1}{2}\|x-z_k\|^2.$$

The set G is convex, so the segment joining $z_k \in G$ and $z_m \in G$ lies in G. In particular, $\frac{z_m+z_k}{2} \in G$.

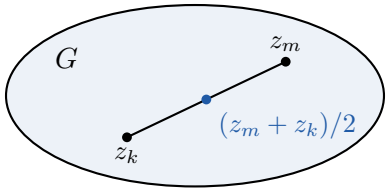

Figure 4: By convexity, the segment from z_k to z_m, and so its midpoint, lies in G.

Since $d = \inf_{z\in G}\|x-z\|$,

$$d^2 \le \left\|x-\frac{z_m+z_k}{2}\right\|^2.$$

So the left-hand side satisfies

$$d^2+\frac{1}{4}\|z_m-z_k\|^2 \le \left\|x-\frac{z_m+z_k}{2}\right\|^2+\frac{1}{4}\|z_m-z_k\|^2,$$

while the right-hand side satisfies

$$\frac{1}{2}\|x-z_m\|^2+\frac{1}{2}\|x-z_k\|^2 \to \frac{d^2}{2}+\frac{d^2}{2} \qquad \text{as } m,k\to\infty.$$

Therefore

$$\lim_{m,k\to\infty}\|z_m-z_k\|^2=0.$$

X is complete, so $z_m \to z \in X$. G is closed, $z_m \in G$ and $z_m \to z$, so $z \in G$ (this holds in metric spaces). Then

$$d=\lim_{m\to\infty}\|x-z_m\|=\|x-z\|.$$

Uniqueness. Assume there is $w \in G$ such that $\|x-w\|=d$. Apply the parallelogram law to $x-z$ and $x-w$:

$$\|x-z-(x-w)\|^2+\|x-z+(x-w)\|^2=2\|x-z\|^2+2\|x-w\|^2,$$

which is equivalent to

$$\|z-w\|^2=2d^2+2d^2-4\left\|x-\frac{z+w}{2}\right\|^2.$$

Since $\frac{z+w}{2} \in G$, we have $\left\|x - \frac{z+w}{2}\right\| \geq d$, so

$$\|z - w\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0.$$

Hence $w = z$. ■

21-640: Functional Analysis Spring 2023

Lecture 3 — January 23, 2023

Prof. Giovanni Leoni *Scribe: Fan Pu Zeng*

1 Orthogonality

Definition (Orthogonal)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Two vectors $x, y \in X$ are *orthogonal*, written $x \perp y$, if $\langle x, y\rangle = 0$.

For every $x \in X$, the set of vectors orthogonal to x is denoted $x^\perp$.

Remark 2. $x^\perp$ is a closed subspace.

It is a subspace: if $y, z \in x^\perp$ and $t, s \in \mathbb{R}$, then by linearity

$$\langle ty + sz, x\rangle = t\underbrace{\langle y, x\rangle}_{=0} + s\underbrace{\langle z, x\rangle}_{=0} = 0.$$

It is closed: let $g(y) = \langle x, y\rangle$. Then

$$x^\perp = \{y \in X : \langle x, y\rangle = 0\} = g^{-1}(\{0\}),$$

and $\{0\}$ is closed. By the Cauchy–Schwarz inequality,

$$|g(y)| \le \|x\| \, \|y\| \,,$$

so g is continuous (it is linear, so $|g(y) - g(y')| = |g(y - y')| \le \|x\| \, \|y - y'\|$). So $x^\perp$ is closed.

For $E \subseteq X$, define

$$E^\perp = \{y \in X : y \perp x \text{ for every } x \in E\} = \bigcap_{x\in E} x^\perp.$$

An intersection of subspaces is a subspace, and an intersection of closed sets is a closed set, so $\bigcap_{x\in E} x^\perp$ is a closed subspace. Hence $E^\perp$ is a closed subspace.

Last time we saw that for $x, y \in X$,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \,.$$

Theorem 3 (Pythagorean theorem)**.** *If $x \perp y$, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \,.$$

2 The Projection Theorem

Theorem 4 (Projection)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $Y \subset X$ be a closed subspace. Then every $x \in X$ has a unique decomposition*

$$x = y + z, \qquad y \in Y, \quad z \in Y^{\perp}.$$

Moreover, the map

$$P : X \to Y, \qquad x \mapsto y,$$

is linear and Lipschitz continuous, and

$$P \circ P = P, \qquad Y = P(X), \qquad Y^{\perp} = (I - P)(X),$$

$$X = Y + Y^{\perp}, \qquad \|x - P(x)\| = \operatorname{dist}(x, Y),$$

where I is the identity.

Remark 3. Here $\subset$ denotes a proper subspace; we use $\subseteq$ for non-strict inclusion.

Proof. Let $x_0 \in X$ and $G = Y$, which is closed and convex. So by the theorem on projections onto closed convex sets (with its unique minimizer), we can find a unique y_0 such that

$$\|x_0 - y_0\| = \min_{w \in Y} \|x_0 - w\| .$$

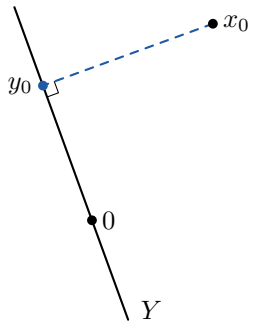

Figure 5: The closest point $y_0 \in Y$ to x_0; the segment from x_0 to y_0 is orthogonal to Y.

Take $t \in \mathbb{R}$ and $y \in Y$. Since $y_0 + ty \in Y$ and y_0 is the minimizer,

$$\|x_0 - y_0\|^2 \leq \|x_0 - (y_0 + ty)\|^2 ,$$

so

$$\|x_0 - y_0\|^2 \leq \|x_0 - y_0\|^2 + t^2 \|y\|^2 - 2t \langle x_0 - y_0, y\rangle ,$$

that is,

$$0 \leq t^2 \|y\|^2 - 2t \langle x_0 - y_0, y\rangle .$$

If $t > 0$, divide by t:

$$0 \leq t \|y\|^2 - 2 \langle x_0 - y_0, y\rangle .$$

Letting $t \to 0^+$ gives $0 \le -2\langle x_0 - y_0, y\rangle$.

If $t < 0$, dividing by t gives

$$0 \ge t\|y\|^2 - 2\langle x_0 - y_0, y\rangle.$$

Letting $t \to 0^-$ gives $0 \ge -2\langle x_0 - y_0, y\rangle$.

So $\langle x_0 - y_0, y\rangle = 0$ for every $y \in Y$, that is, $x_0 - y_0 \in Y^\perp$. Writing $z_0 = x_0 - y_0$,

$$x_0 = \underbrace{y_0}_{\in Y} + \underbrace{x_0 - y_0}_{=z_0 \in Y^\perp}.$$

Uniqueness. Suppose

$$x_0 = y_0 + z_0 = y_1 + z_1, \qquad y_0, y_1 \in Y, \quad z_0, z_1 \in Y^\perp.$$

Then

$$\underbrace{y_0 - y_1}_{\in Y} = \underbrace{z_1 - z_0}_{\in Y^\perp}.$$

But $Y \cap Y^\perp = \{0\}$ (if $v \in Y \cap Y^\perp$ then $\langle v, v\rangle = 0$, so $v = 0$). So $y_0 = y_1$ and $z_0 = z_1$.

Linearity. The map $P : X \to Y$, $x \mapsto y$, is well defined. Write

$$x = y + z = P(x) + z,$$
$$x_1 = y_1 + z_1 = P(x_1) + z_1,$$

where $y, y_1 \in Y$ and $z, z_1 \in Y^\perp$. For $s, t \in \mathbb{R}$, take the linear combination

$$sx + tx_1 = \underbrace{(sy + ty_1)}_{\in Y} + \underbrace{(sz + tz_1)}_{\in Y^\perp}.$$

By uniqueness of the decomposition,

$$P(sx + tx_1) = sP(x) + tP(x_1),$$

so P is a linear function.

Lipschitz continuity.

Claim 2. P is Lipschitz.

Write

$$x = \underbrace{P(x)}_{\in Y} + \underbrace{x - P(x)}_{\in Y^\perp}, \qquad \text{so } P(x) \perp x - P(x).$$

By the Pythagorean theorem,

$$\|x\|^2 = \|P(x)\|^2 + \underbrace{\|x - P(x)\|^2}_{\ge 0} \ge \|P(x)\|^2,$$

so

$$\boxed{\|x\| \ge \|P(x)\|.}$$

By linearity ($P(x) - P(x_1) = P(x - x_1)$),

$$\boxed{\|P(x) - P(x_1)\| \le \|x - x_1\|,}$$

so P is 1-Lipschitz. ∎

3 The Dual Space

Definition (Dual space)**.** Let $(X, \|\cdot\|)$ be a normed space. The *dual space* of X, denoted X', is the space of all $L : X \to \mathbb{R}$ that are linear and continuous. Define the *norm of* L by

$$\|L\|_{X'} := \sup\left\{\frac{|L(x)|}{\|x\|} : x \in X \setminus \{0\}\right\}.$$

Example 5. $(X', \|\cdot\|_{X'})$ is a normed space.

Theorem 5 (Riesz representation theorem)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then for every $L \in X'$ there is a* unique *$y \in X$ such that*

$$L(x) = \langle y, x\rangle \qquad \textit{for all } x \in X.$$

Moreover, $\|L\|_{X'} = \|y\|_X$.

Idea: characterizing the dual of a given space. For every space you like, there will be a Riesz representation theorem.

Remark 4. If $y \in X$, define

$$L_y(x) := \langle y, x\rangle, \qquad x \in X.$$

Then $L_y : X \to \mathbb{R}$ is linear, and

$$|L_y(x)| \le \|y\|\,\|x\|$$

by the Cauchy–Schwarz inequality, so L_y is continuous. This gives a map

$$X \ni y \overset{T}{\longmapsto} L_y \in X'.$$

T is one-to-one, onto, and preserves the norm (an isomorphism between normed spaces). So we can identify X with $T(X) = X'$.

Example 6. Let $E \subseteq \mathbb{R}^N$ be Lebesgue measurable. If $[g] \in L^2(E)$, then

$$L_{[g]}([f]) := \int_E fg\,d\mathbf{x}, \qquad [f] \in L^2(E),$$

is linear and continuous, and

$$\left\|L_{[g]}\right\|_{(L^2(E))'} = \|[g]\|_{L^2(E)}.$$

Conversely, for every $L : L^2(E) \to \mathbb{R}$ linear and continuous, there is a unique $[g] \in L^2(E)$ such that

$$L([f]) = \int_E fg\,d\mathbf{x} \qquad \text{for all } [f] \in L^2(E),$$

and $\|L\|_{(L^2(E))'} = \|[g]\|_{L^2(E)}$.

So $[g]$, an equivalence class of functions, corresponds to $L_{[g]}$, a linear functional on these functions.

Lecture 4 — January 25, 2023

Prof. Giovanni Leoni *Scribe: Fan Pu Zeng*

1 Proof of the Riesz Representation Theorem

Recall the setting: $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $L \in X'$. We want $y_0 \in X$ with $L(x) = \langle x, y_0\rangle$ for every $x \in X$, and $\|L\|_{X'} = \|y_0\|$.

Proof. Step 1: existence. If $L = 0$, take $y_0 = 0$.

Assume $L \neq 0$. Let

$$Y = \ker L = \{x \in X : L(x) = 0\} = L^{-1}(\{0\}).$$

Since L is linear, Y is a subspace of X. Since L is continuous and $\{0\}$ is closed, Y is closed. Since $L \neq 0$, $Y \subset X$ is a proper subspace, so by the projection theorem (which gives $X = Y + Y^\perp$)

$$Y^\perp \neq \{0\}.$$

Fix $z_0 \in Y^\perp$ with $z_0 \neq 0$. Define

$$y_0 := \underbrace{\frac{L(z_0)}{\|z_0\|^2}}_{\text{a real number}} z_0 \in Y^\perp,$$

which lies in $Y^\perp$ because $z_0 \in Y^\perp$ and $Y^\perp$ is a subspace. Since L is linear,

$$L(y_0) = \frac{L(z_0)}{\|z_0\|^2} L(z_0) = \frac{L(z_0)^2}{\|z_0\|^4} \|z_0\|^2 = \langle y_0, y_0\rangle . \tag{$*$}$$

In particular $y_0 \neq 0$: $z_0 \in Y^\perp \setminus \{0\}$ is not in Y, so $L(z_0) \neq 0$.

For every $x \in X$, define

$$x' = x - \frac{L(x)}{\|y_0\|^2} y_0.$$

By linearity and $(*)$,

$$L(x') = L(x) - \frac{L(x)}{\|y_0\|^2} L(y_0) = L(x) - \frac{L(x)}{\|y_0\|^2} \langle y_0, y_0\rangle = 0.$$

So $x' \in \ker L = Y$. But $y_0 \in Y^\perp$, so $x' \perp y_0$, and

$$0 = \langle x', y_0\rangle = \left\langle x - \frac{L(x)}{\|y_0\|^2} y_0,\ y_0 \right\rangle = \langle x, y_0\rangle - \frac{L(x)}{\|y_0\|^2} \langle y_0, y_0\rangle = \langle x, y_0\rangle - L(x).$$

So $L(x) = \langle x, y_0\rangle$ for every $x \in X$.

So every linear and continuous function can be written as an inner product of something with x.

Step 2: the norm. Suppose $y_0 \in X$ and $L(x) = \langle x, y_0 \rangle$ for all $x \in X$. Recall

$$\|L\|_{X'} = \sup \left\{ \frac{|L(x)|}{\|x\|} : x \in X \setminus \{0\} \right\}.$$

By the Cauchy–Schwarz inequality,

$$|L(x)| = |\langle x, y_0 \rangle| \leq \|x\| \, \|y_0\|,$$

so if $x \neq 0$,

$$\frac{|L(x)|}{\|x\|} \leq \|y_0\|,$$

and hence $\|L\|_{X'} \leq \|y_0\|$.

If $y_0 = 0$, we are done. If $y_0 \neq 0$, then

$$\frac{|L(y_0)|}{\|y_0\|} = \frac{|\langle y_0, y_0 \rangle|}{\|y_0\|} = \|y_0\|.$$

So the supremum is a maximum, equal to $\|y_0\|$. ■

We can identify X with X' via

$$y_0 \longleftrightarrow L_{y_0}.$$

Example 7. Let $(X, \|\cdot\|)$ be a normed space and $L : X \to \mathbb{R}$ linear.

(1) Prove that L is continuous if and only if

$$\|L\|_{X'} = \sup_{x \neq 0} \frac{|L(x)|}{\|x\|} < \infty.$$

This gives Lipschitz continuity.

(2) Prove that L is continuous if and only if $\ker L$ is closed.

Example 8. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space. We can find $\{e_m\}_{m \in \mathbb{N}}$ linearly independent. Let

$$Y = \operatorname{span}\{e_m : m \in \mathbb{N}\}.$$

Define $L(e_m) := m \, \|e_m\|$. This gives $L : Y \to \mathbb{R}$ linear. Extend L linearly to $X \setminus Y$. Then

$$\|L\|_{X'} \geq \sup_m \frac{|L(e_m)|}{\|e_m\|} = \sup_m \frac{m \, \|e_m\|}{\|e_m\|} = \infty.$$

So L is not continuous, and therefore $\ker L$ is not closed.

2 Fourier Series

Definition. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

(1) $E \subseteq X$ is an *orthogonal set* if $x \perp y$ for all $x, y \in E$ with $x \neq y$.

(2) $E \subseteq X$ is an *orthonormal set* if E is orthogonal and $\|x\| = 1$ for all $x \in E$. That is, for $x, y \in E$,
$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

(3) An orthonormal set E is *complete* if $X = \overline{\operatorname{span} E}$.

Proposition 1. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and let E be an orthonormal set. Then E is linearly independent. (Exercise.)

Definition (Fourier coefficients)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $E = \{e_\alpha : \alpha \in \Lambda\}$ be orthonormal. For every $x \in X$, the numbers
$$\hat{x}(\alpha) = \langle x, e_\alpha\rangle$$
are the *Fourier coefficients* of x relative to E.

Definition (Infinite sums)**.** Let Λ be a set (it can be uncountable), and let $x_\alpha \geq 0$ for $\alpha \in \Lambda$. The (infinite) sum is
$$\sum_{\alpha\in\Lambda} x_\alpha := \sup\left\{\sum_{\alpha\in F} x_\alpha : F \subseteq \Lambda,\ F \text{ finite}\right\}.$$

Example 9. If $\sum_{\alpha\in\Lambda} x_\alpha < \infty$, prove that $\{\alpha \in \Lambda : x_\alpha > 0\}$ is countable.

(Key idea: the infinite sum is defined as the supremum over all finite sums.)

Theorem 6 (Bessel's inequality)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $E = \{e_\alpha : \alpha \in \Lambda\}$ be orthonormal. Then for every $x \in X$,*
$$\sum_{\alpha\in\Lambda} |\hat{x}(\alpha)|^2 \leq \|x\|^2,$$
where $\hat{x}(\alpha) = \langle x, e_\alpha\rangle$ and Λ can be uncountable.

Proof. Let $F \subseteq E$ be finite, $F = \{e_1, e_2, \ldots, e_m\}$. Then
$$0 \leq \left\| x - \sum_{i=1}^m \langle x, e_i\rangle\, e_i \right\|^2 = \|x\|^2 - 2\left\langle x, \sum_{i=1}^m \langle x, e_i\rangle\, e_i \right\rangle + \left\| \sum_{i=1}^m \langle x, e_i\rangle\, e_i \right\|^2.$$
Since $e_i \perp e_j$ for $i \neq j$, the Pythagorean theorem gives
$$\left\| \sum_{i=1}^m \langle x, e_i\rangle\, e_i \right\|^2 = \sum_{i=1}^m \|\langle x, e_i\rangle\, e_i\|^2 = \sum_{i=1}^m |\langle x, e_i\rangle|^2.$$

By linearity,

$$\left\langle x, \sum_{i=1}^{m} \langle x, e_i \rangle \, e_i \right\rangle = \sum_{i=1}^{m} \langle x, e_i \rangle^2 .$$

Plugging these back in,

$$0 \leq \|x\|^2 - 2 \sum_{i=1}^{m} \langle x, e_i \rangle^2 + \sum_{i=1}^{m} \langle x, e_i \rangle^2 ,$$

so

$$\sum_{i=1}^{m} \langle x, e_i \rangle^2 \leq \|x\|^2 .$$

This holds for every finite $F = \{e_1, \ldots, e_m\} \subseteq E$. Taking the supremum over all finite sets F,

$$\sum_{\alpha \in \Lambda} \langle x, e_\alpha \rangle^2 \leq \|x\|^2 . \qquad \blacksquare$$

In general, we only have equality when E is complete.

In L^2, Bessel's inequality reads

$$\sum_{\alpha \in \Lambda} \left(\int_E f(\mathbf{x}) \, e_\alpha(\mathbf{x}) \, d\mathbf{x} \right)^2 \leq \int_E |f(\mathbf{x})|^2 \, d\mathbf{x}.$$

Lecture 5 — January 27, 2023

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1 Recall: Orthonormal Sets

Last time we saw orthogonal and orthonormal sets.

Example 10. If $\sum_{\alpha\in\Lambda} y_\alpha < \infty$, then $\{\alpha \in \Lambda : y_\alpha > 0\}$ is countable.

Combined with Bessel's inequality (applied with $y_\alpha = |\langle x, e_\alpha\rangle|^2$), this gives the punchline:

Punchline. $\langle x, e_\alpha\rangle = 0$ for all but countably many $\alpha \in \Lambda$.

2 Fourier Series

Proposition 2. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $E \subseteq X$ be nonempty. Then

$$\overline{\operatorname{span} E} = X \quad \text{(that is, } \operatorname{span} E \text{ is dense)} \qquad \text{if and only if} \qquad E^\perp = \{0\}.$$

Proof. ($\Rightarrow$) Assume $\overline{\operatorname{span} E} = X$. Let $x \in E^\perp \subseteq X = \overline{\operatorname{span} E}$, so $x \in \overline{\operatorname{span} E}$. So we can find $x_m \in \operatorname{span} E$ with $x_m \to x$. Each x_m is a linear combination of elements of E, so $\langle x_m, x\rangle$ is a linear combination of terms $\langle \text{element of } E, x\rangle = 0$. Hence

$$0 = \langle x_m, x\rangle \to \langle x, x\rangle = \|x\|^2,$$

so $x = 0$.

($\Leftarrow$) Assume $E^\perp = \{0\}$. Let $x \in X$ with $x \perp \operatorname{span} E$. Since $\operatorname{span} E \supseteq E$, we get $x \perp E$, so $x \in E^\perp = \{0\}$. So $x = 0$, and

$$(\operatorname{span} E)^\perp = \{0\}.$$

Let $Y = \overline{\operatorname{span} E}$, a closed subspace. By continuity of the inner product, $Y^\perp = (\operatorname{span} E)^\perp = \{0\}$, so $X = Y + Y^\perp = Y$ by the projection theorem. So $Y = X$. ∎

Theorem 7 (Fourier series)**.** *Let X be a Hilbert space and let $E = \{e_\alpha : \alpha \in \Lambda\}$ be orthonormal. The following are equivalent:*

(1) E is maximal with respect to inclusion.

(2) E is complete, that is, $\overline{\operatorname{span} E} = X$.

(3) Parseval's identity*: for every $x \in X$,*

$$\sum_{\alpha\in\Lambda} |\langle x, e_\alpha\rangle|^2 = \|x\|^2.$$

Proof. (1) $\Rightarrow$ (2). Assume E is a maximal orthonormal set. Since $\operatorname{span} E$ is a subspace, $\overline{\operatorname{span} E}$ is a closed subspace. If $\overline{\operatorname{span} E} \neq X$, then by the projection theorem

$$\left(\overline{\operatorname{span} E}\right)^{\perp} \neq \{0\}.$$

Let $x \in \left(\overline{\operatorname{span} E}\right)^{\perp} \setminus \{0\}$. Then

$$E \cup \left\{ \frac{x}{\|x\|} \right\}$$

is orthonormal, which contradicts the fact that E is maximal.

(2) $\Rightarrow$ (1). Assume E is complete, that is, $\overline{\operatorname{span} E} = X$. By the proposition, $E^{\perp} = \{0\}$, so E is maximal (a larger orthonormal set would contain a unit vector in $E^{\perp}$).

(3) $\Rightarrow$ (1). Let $x \in X$ with $x \perp E$. Then $\langle x, e_\alpha \rangle = 0$ for all $\alpha \in \Lambda$. By Parseval's identity,

$$\sum_{\alpha \in \Lambda} 0 = \|x\|^2,$$

so $x = 0$. So $E^{\perp} = \{0\}$, and E is maximal.

(1) $\Rightarrow$ (3). Assume E is maximal. By Bessel's inequality,

$$\sum_{\alpha} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2 < \infty.$$

By the punchline, the set $\{e_\alpha : \langle x, e_\alpha \rangle \neq 0\} = \{e_m : m \in \mathbb{N}\}$ is countable (or finite), and

$$\sum_{m=1}^{\infty} |\langle x, e_m \rangle|^2 \leq \|x\|^2 < \infty. \tag{$*$}$$

For $\ell \in \mathbb{N}$, let

$$x_\ell := \sum_{m=1}^{\ell} \langle x, e_m \rangle e_m.$$

For $k > \ell$, since $\|e_m\| = 1$ and the e_m are orthogonal (Pythagorean theorem),

$$\|x_k - x_\ell\|^2 = \left\| \sum_{m=\ell+1}^{k} \langle x, e_m \rangle e_m \right\|^2 = \sum_{m=\ell+1}^{k} \langle x, e_m \rangle^2 \xrightarrow[\ell \to \infty]{} 0$$

by $(*)$. So $\{x_\ell\}_\ell$ is a Cauchy sequence, and X is Hilbert, so $x_\ell \to y \in X$, where

$$y = \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m$$

is the *Fourier series* of x.

Claim 3. $x - y \perp E$. (Since E is maximal, this gives $x = y$.)

Proof of the claim. Fix k and take $\ell > k$. Then

$$\begin{aligned}\langle x - x_\ell, e_k\rangle &= \left\langle x - \sum_{m=1}^{\ell} \langle x, e_m\rangle\, e_m,\ e_k \right\rangle \\ &= \langle x, e_k\rangle - \sum_{m=1}^{\ell} \langle x, e_m\rangle \langle e_m, e_k\rangle \\ &= \langle x, e_k\rangle - \langle x, e_k\rangle = 0,\end{aligned}$$

since

$$\langle e_m, e_k\rangle = \delta_{m,k} = \begin{cases} 1 & \text{if } m = k, \\ 0 & \text{otherwise.}\end{cases}$$

Letting $\ell \to \infty$ in $\langle x - x_\ell, e_k\rangle = 0$ gives $\langle x - y, e_k\rangle = 0$. So $x - y \perp e_k$ for all k.

If $\alpha \in \Lambda$ is such that $\langle x, e_\alpha\rangle = 0$, then

$$\langle x - x_\ell, e_\alpha\rangle = \underbrace{\langle x, e_\alpha\rangle}_{=0} - \sum_{m=1}^{\ell} \langle x, e_m\rangle \underbrace{\langle e_m, e_\alpha\rangle}_{=0} = 0,$$

and letting $\ell \to \infty$ gives $\langle x - y, e_\alpha\rangle = 0$.

So $x - y \perp E$. Since E is maximal, $x = y$ (otherwise $E \cup \{(x-y)/\|x-y\|\}$ would be a larger orthonormal set).

Now for the finite sums: expanding as in the proof of Bessel's inequality,

$$\underbrace{\left\| x - \sum_{m=1}^{\ell} \langle x, e_m\rangle\, e_m \right\|^2}_{\to 0 \text{ as } \ell\to\infty} = \|x\|^2 - \sum_{m=1}^{\ell} \langle x, e_m\rangle^2 .$$

The left-hand side goes to 0 (because $x_\ell \to y = x$), so the right-hand side goes to 0, and then

$$\|x\|^2 = \sum_{m=1}^{\infty} \langle x, e_m\rangle^2 + 0 = \sum_{\alpha} \langle x, e_\alpha\rangle^2 . \qquad \blacksquare$$

1 Existence of Complete Orthonormal Sets

Theorem 8. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $E \subseteq X$ be orthonormal. Then there is $F \supseteq E$ that is orthonormal and complete.*

Proof. Let $\mathcal{O}$ be the family of all orthonormal sets of X that contain E. Inclusion is a partial order on $\mathcal{O}$.

Let $\mathcal{G}$ be a totally ordered subset of $\mathcal{O}$,

$$\mathcal{G} = \{E_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{O}.$$

That is, if $E_\alpha, E_\beta \in \mathcal{G}$, then $E_\alpha \subseteq E_\beta$ or $E_\beta \subseteq E_\alpha$. Define

$$G = \bigcup_{\alpha\in\Lambda} E_\alpha \subseteq X.$$

Since each $E_\alpha \supseteq E$, we have $G \supseteq E$, and $G \supseteq E_\alpha$ for every $\alpha \in \Lambda$.

Claim 4. G is orthonormal.

If $x, y \in G$, then $x \in E_\alpha$ and $y \in E_\beta$ for some $\alpha, \beta \in \Lambda$. So $E_\alpha \subseteq E_\beta$ or $E_\beta \subseteq E_\alpha$. Say $E_\alpha \subseteq E_\beta$. Then $x, y \in E_\beta$, which is orthonormal, so

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

So $G \in \mathcal{O}$, and G is an upper bound of $\mathcal{G}$. By Zorn's lemma, $\mathcal{O}$ admits a maximal element F. It satisfies $F \supseteq E$ and is orthonormal.

Claim 5. F is complete. By the Fourier series theorem, this is equivalent to F being maximal with respect to inclusion.

Let $x \in X$ with $x \perp F$. If $x \neq 0$, then

$$F \cup \left\{\frac{x}{\|x\|}\right\} \supseteq E$$

is orthonormal and larger than F. This contradicts the maximality of F in $\mathcal{O}$. So $x = 0$, that is, $F^\perp = \{0\}$, and F is complete. ∎

Last time we showed that if F is complete and $x \in X$, then

$$x = \sum_{e \in F} \langle x, e \rangle \, e \qquad \text{(Fourier series)},$$

$$\|x\|^2 = \sum_{e \in F} \langle x, e \rangle^2 \qquad \text{(Parseval's identity)}.$$

We are done with Hilbert spaces; now we go to Banach spaces.

2 Banach Spaces

Definition (Norm)**.** Let X be a vector space. A *norm* is a function $\|\cdot\| : X \to [0, \infty)$ such that

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|tx\| = |t| \, \|x\|$ for all $t \in \mathbb{R}$, $x \in X$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Then $d(x, y) = \|x - y\|$ is a metric.

Definition (Banach space)**.** A normed space $(X, \|\cdot\|)$ is a *Banach space* if (X, d) is a complete metric space.

Definition (Dual space)**.** Let $(X, \|\cdot\|)$ be a normed space. The *dual space* X' of X is the space of all $L : X \to \mathbb{R}$ that are linear and continuous, with

$$\|L\|_{X'} := \sup_{x \neq 0} \frac{|L(x)|}{\|x\|}.$$

Example 11. Let $(X, \|\cdot\|)$ be a normed space. Show that

(1) $\|\cdot\|_{X'}$ is a norm;

(2) $(X', \|\cdot\|_{X'})$ is a Banach space (even if X is not complete).

2.1 Examples

(1) $\mathbb{R}^N$ with $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

(2) Every inner product space is a normed space. Every Hilbert space is a Banach space.

(3) Let $1 \leq p < \infty$ and let $E \subseteq \mathbb{R}^N$ be Lebesgue measurable. The space $L^p(E)$ is the set of all equivalence classes $[f]$, where $f : E \to \mathbb{R}$ is measurable with

$$\int_E |f(\mathbf{x})|^p \, d\mathbf{x} < \infty,$$

with the norm

$$\|[f]\|_{L^p(E)} = \left(\int_E |f(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p}.$$

$\boxed{L^p(E) \text{ is a Banach space.}}$

Next we look at its dual, which is $L^q(E)$, also a Banach space.

2.2 The dual of $L^p(E)$ for $1 < p < \infty$

Assume $1 < p < \infty$. Let $1 < q < \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

the *Hölder conjugate exponent* of p.

Hölder's inequality. If $[f] \in L^p(E)$ and $[g] \in L^q(E)$, then

$$\left| \int_E f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x} \right| \leq \|[f]\|_{L^p(E)}\, \|[g]\|_{L^q(E)}\,.$$

Recall that $\|L\|_{X'} = \sup_{x \neq 0} |L(x)| / \|x\|$. If $[f] \neq [0]$, then

$$\frac{|\int_E f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}|}{\|[f]\|_{L^p(E)}} \leq \|[g]\|_{L^q(E)}\,,$$

which shows continuity. Taking the supremum,

$$\sup_{[f]\neq[0]} \frac{|\int_E f g\, d\mathbf{x}|}{\|[f]\|_{L^p(E)}} \leq \|[g]\|_{L^q(E)}\,.$$

So

$$L_{[g]} : L^p(E) \to \mathbb{R}, \qquad [f] \mapsto \int_E f g\, d\mathbf{x},$$

is linear and continuous, and

$$\left\| L_{[g]} \right\|_{(L^p(E))'} \leq \|[g]\|_{L^q(E)}\,.$$

In fact equality holds (exercise).

Theorem 9 (Riesz representation in L^p)**.** *Let $E \subseteq \mathbb{R}^N$ be measurable and $1 < p < \infty$. Then for every $L \in (L^p(E))'$ there is a unique $[g] \in L^q(E)$ such that*

$$L([f]) = \int_E f g\, d\mathbf{x} \qquad \textit{for all } [f] \in L^p(E),$$

and

$$\|L\|_{(L^p(E))'} = \|[g]\|_{L^q(E)}\,.$$

So the map

$$T : L^q(E) \to (L^p(E))', \qquad [g] \mapsto L_{[g]},$$

is one-to-one, onto, and preserves the norm. We identify

$$[g] \longleftrightarrow L_{[g]},$$

that is, we identify a function with a linear function.

Thus, for $1 < p < \infty$ and $1 < q < \infty$,

$$(L^p(E))' \cong L^q(E),$$

and so

$$(L^p(E))'' = ((L^p(E))')' \cong (L^q(E))' \cong L^p(E).$$

A space with $X'' \cong X$ is a *reflexive space*: for $1 < p < \infty$, $L^p(E)$ is reflexive.

2.3 The case $p = 1$

Since $\frac{1}{p} + \frac{1}{q} = 1$, for $p = 1$ we get $q = \infty$. Define

$$L^\infty(E) = \left\{ [f] : f : E \to \mathbb{R} \text{ measurable and } \operatorname*{ess\,sup}_E |f| < \infty \right\},$$

where the *essential supremum* is

$$\operatorname*{ess\,sup}_E g := \inf\{L : g(\mathbf{x}) \le L \text{ for a.e. } \mathbf{x} \in E\}.$$

Then

$$\|[f]\|_{L^\infty(E)} := \operatorname*{ess\,sup}_E |f|$$

is a norm (exercise), and $L^\infty(E)$ is complete.

Similarly,

$$(L^1(E))' \cong L^\infty(E),$$

$$\left| \int_E f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x} \right| \le \operatorname*{ess\,sup}_E |g| = \|[f]\|_{L^1(E)}.$$

But L^1 is not reflexive.