21-329: Set Theory

Spring 2023 • Dr. Benjamin Siskind

Notes by Fan Pu Zeng

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Lecture 1 — February 27, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Associativity of Ordinal Addition: A Second Proof

Let α, β, γ be ordinals. We want to show

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

By definition of ordinal addition,

$$(\alpha + \beta) + \gamma = \text{type}\big(((\alpha, \in) \frown (\beta, \in)) \frown (\gamma, \in)\big),$$
$$\alpha + (\beta + \gamma) = \text{type}\big((\alpha, \in) \frown ((\beta, \in) \frown (\gamma, \in))\big).$$

So it is enough to show

$$((\alpha, \in) \frown (\beta, \in)) \frown (\gamma, \in) \;\cong\; (\alpha, \in) \frown ((\beta, \in) \frown (\gamma, \in)).$$

This time we make use of the structure, instead of the previous mechanical proof.

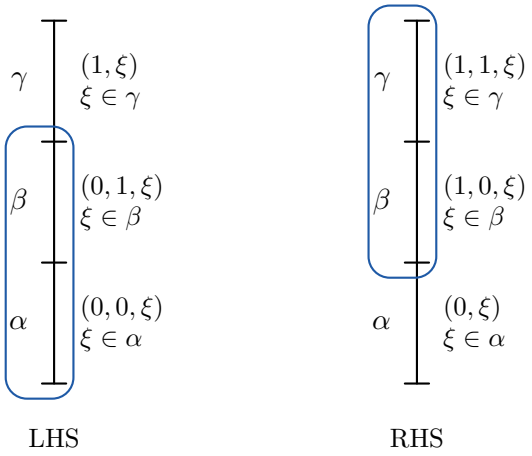

Figure 1: The two concatenations, with each element labelled by its tuple; the circled part is the inner concatenation.

Define an isomorphism π from the left-hand side to the right-hand side as follows:

- $\pi(0, 0, \xi) = (0, \xi)$ for $\xi \in \alpha$;
- $\pi(0, 1, \xi) = (1, 0, \xi)$ for $\xi \in \beta$;
- $\pi(1, \xi) = (1, 1, \xi)$ for $\xi \in \gamma$.

On both sides, every element of the α-part comes before every element of the β-part, which comes before every element of the γ-part, and within each part the order is that of α, β or γ. So π is an order-preserving bijection.

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Lecture 2 — March 1, 2023

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1 Last Time

We saw more on ordinal addition, for example associativity: for all α, β, γ,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

2 Ordinal Subtraction

Here is one more useful fact.

Definition. For ordinals α, β,

$$\alpha < \beta \iff \alpha \in \beta,$$
$$\alpha \leq \beta \iff \alpha < \beta \text{ or } \alpha = \beta.$$

Proposition 1. If $\alpha \leq \beta$, then there is a unique ordinal γ such that $\alpha + \gamma = \beta$.

We can think of this as a form of subtraction.

Fact 1. For any α, β, we have $\alpha + \beta \geq \beta$.

Proof. By induction on β; this is left as an informal exercise. ∎

Proof of Proposition 1. **Existence.** Look at the set

$$S = \{\gamma \leq \beta \mid \alpha + \gamma \geq \beta\}.$$

By Fact 1, $\beta \in S$, so S is nonempty. Fix

$$\gamma = \text{the least element of } S = \min(S).$$

We know $\alpha + \gamma \geq \beta$, so we just need to see that $\alpha + \gamma \not> \beta$.

Case 1: $\gamma = 0$. Then $\alpha = \alpha + 0 \geq \beta$. But $\alpha \leq \beta$ too, so $\alpha = \beta$, that is, $\alpha + 0 = \beta$.

Case 2: $\gamma = \delta + 1$ *for some* δ. Then $\delta \notin S$, as $\gamma > \delta$ and $\gamma = \min(S)$. So

$$\alpha + \delta < \beta \leq (\alpha + \delta) + 1.$$

But $\beta > \alpha + \delta$ implies $\beta \geq (\alpha + \delta) + 1$. So

$$\beta = (\alpha + \delta) + 1 = \alpha + \gamma.$$

Case 3: γ is a limit ordinal. Then

$$\alpha + \gamma = \sup\{\alpha + \delta \mid \delta \in \gamma\} = \bigcup\{\alpha + \delta \mid \delta \in \gamma\}.$$

For any $\delta \in \gamma$ we have $\delta \notin S$, so $\alpha + \delta < \beta$. Hence $\bigcup\{\alpha + \delta \mid \delta \in \gamma\} \leq \beta$, and so $\beta = \alpha + \gamma$.

(This is a case analysis, not an induction.)

Uniqueness. Recall from last time that whenever $\eta < \xi$,

$$\alpha + \eta < \alpha + \xi,$$

that is, addition is strictly increasing in the second coordinate. So if $\alpha + \gamma = \beta$, then for any $\delta \neq \gamma$ we have $\alpha + \delta \neq \beta$, because either

$$\alpha + \delta < \alpha + \gamma = \beta \qquad \text{or} \qquad \alpha + \delta > \alpha + \gamma = \beta.$$ ■

Definition. For $\alpha \leq \beta$, let $\beta - \alpha$ be the unique γ such that $\alpha + \gamma = \beta$.

Remark 1 (Warning)**.** $\beta - \alpha$ is not the set difference $\{\gamma \in \beta \mid \gamma \notin \alpha\}$. But

$$(\{\gamma \in \beta \mid \gamma \notin \alpha\}, \in) \cong \beta - \alpha.$$

We can do a proof by pictures again:

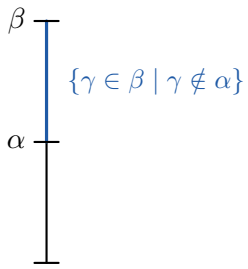

Figure 2: The ordinals below β; the highlighted ones, those not below α, form a well-order of type $\beta - \alpha$.

Remark 2 (Warning 2)**.** In general, it is possible to have $\beta - \alpha = \beta$.

Example 1. $1 + \omega = \omega$, so $\omega - 1 = \omega$.

So this may cause problems in induction if you take differences, as you may not actually get a smaller ordinal.

3 Ordinal Multiplication

Definition (Recursive definition of ordinal multiplication)**.**

- $\alpha \cdot 0 = 0$;
- $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$;
- $\alpha \cdot \lambda = \sup\{\alpha \cdot \beta \mid \beta \in \lambda\}$ when λ is a limit.

3.1 Products of well-orders

Definition. For $(A, <_A)$ and $(B, <_B)$, define

$$(A, <_A) * (B, <_B) = (C, <),$$

where $C = A \times B$ and $<$ is $<_{\text{lex}}$ (which depends on $<_A$ and $<_B$):

$$(a, b) <_{\text{lex}} (a', b') \iff (a = a' \text{ and } b <_B b') \text{ or } a <_A a'.$$

That is, we replace the elements of A by copies of B.

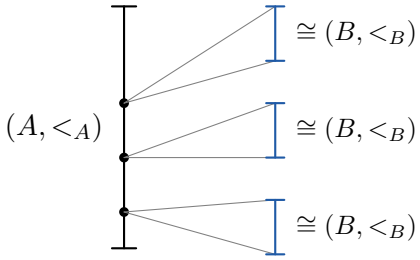

Figure 3: $(A, <_A) * (B, <_B)$: each element of A is replaced by a copy of B.

Lemma 1. *$(A, <_A) * (B, <_B)$ is a well-ordering.*

Partial proof. We'll check irreflexivity and well-foundedness.

Irreflexivity. We want to show $(a, b) \not<_{\text{lex}} (a, b)$. Since $(B, <_B)$ is irreflexive, $b \not<_B b$, and $a \not<_A a$ since $(A, <_A)$ is irreflexive.

Well-foundedness. Let $S \subseteq A \times B$ be nonempty. Let

$$S_0 = \{a \in A \mid \exists b\ (a, b) \in S\}.$$

Since $S_0 \subseteq A$ is nonempty and $(A, <_A)$ is a well-order, let a_0 be the $<_A$-least element of S_0. Let

$$S_1 = \{b \in B \mid (a_0, b) \in S\},$$

and let b_0 be the $<_B$-least element of S_1.

We check that (a_0, b_0) is the $<_{\text{lex}}$-least element of S. Let $(a, b) \in S$. Then $a_0 \leq_A a$ by the choice of a_0. If $a_0 <_A a$, we are done. So suppose $a_0 = a$. Then $(a_0, b) \in S$, so $b \in S_1$, and hence $b_0 \leq_B b$. So $(a_0, b_0) \leq_{\text{lex}} (a, b)$.

(We skipped totality and transitivity.) ■

Theorem 1. *For any α, β,*

$$\alpha \cdot \beta = \text{type}\big((\beta, \in) * (\alpha, \in)\big),$$

that is, $\alpha \cdot \beta \cong (\beta \times \alpha, <_{\text{lex}})$.

Proof. By induction on β.

Base case: $\beta = 0$. Both sides are the empty well-order.

Successor case. We have

$$\alpha \cdot (\beta + 1) \stackrel{\text{def}}{=} \alpha \cdot \beta + \alpha \cong \big((\beta + 1) \times \alpha, <_{\text{lex}}\big).$$

By induction, $\alpha \cdot \beta \cong (\beta \times \alpha, <_{\text{lex}})$, which is β copies of α. So $\alpha \cdot \beta + \alpha$ is β copies of α followed by one more copy of α, which is isomorphic to $\beta + 1$ copies of α. ■

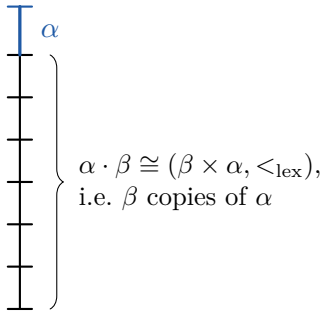

Figure 4: $\alpha \cdot (\beta + 1)$: β copies of α, followed by one more copy of α on top.

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Lecture 3 — March 3, 2023

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1 Ordinal Multiplication as a Product of Well-Orders

Theorem 2. *For any α, β,*

$$\alpha \cdot \beta = \text{type}\big((\beta, \in) * (\alpha, \in)\big) \cong (\beta \times \alpha, <_{\text{lex}}).$$

Proof. By induction on β.

Base case. Trivial.

Successor case. The induction hypothesis is $\alpha \cdot \beta = \text{type}((\beta, \in) * (\alpha, \in))$. We want to show

$$\alpha \cdot (\beta + 1) = \text{type}\big((\beta + 1, \in) * (\alpha, \in)\big).$$

By the theorem on addition,

$$\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha = \text{type}\big((\alpha \cdot \beta, \in) \frown (\alpha, \in)\big).$$

By the induction hypothesis, $(\alpha \cdot \beta, \in) \cong (\beta, \in) * (\alpha, \in)$. So it suffices to show

$$\big((\beta, \in) * (\alpha, \in)\big) \frown (\alpha, \in) \;\cong\; (\beta + 1, \in) * (\alpha, \in).$$

The underlying set of the left-hand side is

$$S = \{(0, \gamma, \delta) \mid \gamma \in \beta \wedge \delta \in \alpha\} \cup \{(1, \delta) \mid \delta \in \alpha\}.$$

The underlying set of the right-hand side is

$$\begin{aligned} T = (\beta + 1) \times \alpha &= (\beta \times \alpha) \cup (\{\beta\} \times \alpha) \\ &= \{(\gamma, \delta) \mid \gamma \in \beta \wedge \delta \in \alpha\} \cup \{(\beta, \delta) \mid \delta \in \alpha\}. \end{aligned}$$

Define $\pi \colon S \to T$ by

- $\pi(0, \gamma, \delta) = (\gamma, \delta)$;
- $\pi(1, \delta) = (\beta, \delta)$.

Checking that π preserves order is an informal exercise.

Limit case. Let λ be a limit. The induction hypothesis is that for all $\beta \in \lambda$,

$$\alpha \cdot \beta = \text{type}\big((\beta, \in) * (\alpha, \in)\big).$$

We want to show

$$\alpha \cdot \lambda = \mathrm{type}\big((\lambda, \in) * (\alpha, \in)\big).$$

On the left-hand side, $\alpha \cdot \lambda = \bigcup\{\alpha \cdot \beta \mid \beta \in \lambda\}$. On the right-hand side, $\lambda \times \alpha = \bigcup\{\beta \times \alpha \mid \beta \in \lambda\}$.

The induction hypothesis gives, for each $\beta \in \lambda$, an isomorphism

$$\pi_\beta \colon (\alpha \cdot \beta, \in) \cong (\beta, \in) * (\alpha, \in).$$

Fact 2. If $\beta \in \gamma \in \lambda$, then $\pi_\gamma \restriction (\alpha \cdot \beta) = \pi_\beta$.

Let

$$\pi = \bigcup\{\pi_\beta \mid \beta \in \lambda\}.$$

By Fact 2, π is a function. Moreover,

$$\mathrm{dom}(\pi) = \alpha \cdot \lambda, \qquad \mathrm{ran}(\pi) = \lambda \times \alpha.$$

Since the restriction of π to $\alpha \cdot \beta$ is π_β for each $\beta \in \lambda$, it follows that π is an isomorphism from $\alpha \cdot \lambda$ to $(\lambda, \in) * (\alpha, \in)$. ■

Fact 3.

(1) For all α, β, γ: $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

(2) For all α, β, γ: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

2 Ordinal Exponentiation

Definition (Ordinal exponentiation)**.**

- $\alpha^0 = 1$;
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$;
- $\alpha^\lambda = \sup\{\alpha^\beta \mid \beta \in \lambda\}$ for λ a limit.

3 Cardinals

This begins Chapter 4.

Definition (Cardinality)**.** $A \approx B$, read "A has the same cardinality as B", if and only if there is a bijection from A to B.

The following is the significance of the Axiom of Choice (AC).

Theorem 3. *For every set A there is an ordinal γ such that $A \approx \gamma$.*

It follows that for every set A there is a relation $<$ on A such that $(A, <)$ is a well-ordering. (This heavily uses AC.)

Proof. Assume A is nonempty. Informally:

- Pick $\pi(0) \in A$.
- If $A - \{\pi(0)\} \neq \emptyset$, pick $\pi(1) \in A - \{\pi(0)\}$.

  $\vdots$
- If we've defined $\pi \restriction \alpha$ and $A - \pi[\alpha] \neq \emptyset$, pick $\pi(\alpha) \in A - \pi[\alpha]$.

Formally, let

$$\mathcal{F} = \{X \subseteq A \mid X \neq \emptyset\} = \mathcal{P}(A) - \{\emptyset\}.$$

By AC, let c be a choice function for $\mathcal{F}$, that is,

- $\mathrm{dom}(c) = \mathcal{F}$;
- for all $X \in \mathcal{F}$, $c(X) \in X$.

So:

- Set $\pi(0) = c(A)$.
- If $A - \{\pi(0)\} \neq \emptyset$, set $\pi(1) = c(A - \{\pi(0)\})$.
- In general, if $\pi \restriction \alpha$ is defined and $A - \pi[\alpha] \neq \emptyset$, let $\pi(\alpha) = c(A - \pi[\alpha])$.

We want to show that there is a γ with $A - \pi[\gamma] = \emptyset$. Suppose not, so for all γ, $A - \pi[\gamma] \neq \emptyset$. Consider instead π^{-1}. Since $\pi(\alpha) \notin \pi[\alpha]$ for every α, π is injective, so π^{-1} describes a function with domain A. So by Replacement, π^{-1} has a set as its range. But this is a contradiction, as every γ satisfies

$$\gamma = \pi^{-1}(c(A - \pi[\gamma])),$$

so the range of π^{-1} would contain every ordinal, and there is no set of all ordinals.

Take γ least such that $A - \pi[\gamma] = \emptyset$. Then check that $\pi \colon \gamma \to A$ is a bijection. ■

Definition (Cardinal)**.** An ordinal κ is a *cardinal* if and only if for all $\eta \in \kappa$, $\eta \not\approx \kappa$.

So κ is like a new size: it is bigger than the things in it.

Definition. For A a set,

$$|A| = \text{the least } \gamma \text{ such that } A \approx \gamma.$$

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Lecture 4 — March 13, 2023

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1 Cardinality

This is Chapter 4.

Definition. $A \approx B$ if and only if there exists a bijection $f\colon A \to B$.

Observation. $\approx$ is reflexive, symmetric and transitive.

Lemma 2 (uses AC)**.** *For any set A, there is an ordinal γ such that $A \approx \gamma$.*

Equivalently, for every set A there is a relation $<$ on A such that $(A, <)$ is a well-ordering.

Definition. The *cardinality* of A is the least γ such that $A \approx \gamma$. We write $|A| = \gamma$.

Definition. An ordinal κ is a *cardinal* if and only if for all $\eta < \kappa$, $\eta \not\approx \kappa$.

Proposition 2. For any A, $|A|$ is a cardinal.

Proof. Let $\kappa = |A|$, so there is a bijection $f\colon \kappa \to A$. Suppose towards a contradiction that $\eta < \kappa$ and $\eta \approx \kappa$. Let $g\colon \eta \to \kappa$ be a bijection. But then $f \circ g\colon \eta \to A$ is a bijection, contradicting the minimality of $\kappa = |A|$. ■

Proposition 3. For any A, B,

$$A \approx B \iff |A| = |B|.$$

Proof. Let $\kappa = |A|$ and $\lambda = |B|$.

- If $A \approx B$, then $\kappa \approx A$ and $\lambda \approx B$. So $\kappa \approx \lambda$, which implies $\kappa = \lambda$ since both are cardinals.
- If $\lambda = \kappa$, then $\kappa \approx A$ and $\kappa \approx B$, so $A \approx B$. ■

Definition.

- A is *countable* if A is finite or $|A| = \omega$ (that is, $|A| \leq \omega$).
- A is *countably infinite* if $|A| = \omega$.
- A is *uncountable* if A is not countable.

Remark 3.

- Every $n \in \omega$ is a cardinal.

- ω is a cardinal.
- $\omega + 1$ is not a cardinal: $|\omega + 1| = \omega$. Indeed, $\omega + 1 \approx \omega$ via $\pi \colon \omega \to \omega + 1$ with

$$\pi(0) = \omega, \qquad \pi(n+1) = n.$$

- $\omega \cdot \omega$ is countable.

Why is $\omega \cdot \omega$ countable? It is enough to produce a bijection $\pi \colon S \to \omega \cdot \omega$ with $S \subseteq \omega$ and S infinite. This is because if $S \subseteq \omega$ is infinite, the increasing enumeration of S, $\{s_i \mid i \in \omega\}$ where $s_i < s_j$ for all $i < j$, is a bijection between S and ω.

Example 2 (Informal exercise)**.** Let

$$S = \{2^{n+1}3^{m+1} \mid n, m \in \omega\},$$

and let $\pi(2^{n+1}3^{m+1}) = \omega \cdot n + m$. Check that $\pi \colon S \to \omega \cdot \omega$ is a bijection.

2 Uncountable Ordinals

Theorem 4 (Cantor)**.** *Not every ordinal is countable, that is, there is an uncountable ordinal.*

This follows from the next theorem.

Theorem 5 (Cantor's diagonal argument)**.** *For any set A, there is no surjection from A to $\mathcal{P}(A)$.*

Proof. Let $f \colon A \to \mathcal{P}(A)$. Let

$$C = \{x \in A \mid x \notin f(x)\}.$$

Then $C \subseteq A$, so $C \in \mathcal{P}(A)$. We'll show $C \notin f[A]$, so f is not surjective.

Notice that for any $x \in A$, $x \in C \iff x \notin f(x)$. So $C \neq f(x)$ for any $x \in A$. But $f[A] = \{f(x) \mid x \in A\}$, so $C \notin f[A]$. ■

Proof of Theorem 4. (This uses the Axiom of Choice and the fact that $|\mathcal{P}(\omega)|$ is defined.) Let $\kappa = |\mathcal{P}(\omega)|$. By Theorem 5, there is no bijection from ω to $\mathcal{P}(\omega)$, so there is no bijection from ω to $|\mathcal{P}(\omega)| = \kappa$. So $\kappa \neq \omega$. But κ is an infinite ordinal, so $\kappa \geq \omega$, which gives $\kappa > \omega$. ■

Another proof (without AC)

There is a set A of all countable ordinals, and A is actually an ordinal. So A must be the first uncountable ordinal.

Fact 4. An ordinal α is countably infinite if and only if $\alpha = \text{type}(\omega, <)$ for some well-ordering $<$ of ω.

Let

$$\begin{aligned} B &= \{\alpha \mid \alpha = \text{type}(\omega, <) \text{ for } < \text{ a well-ordering of } \omega\} \\ &= \text{the set of countably infinite ordinals.} \end{aligned}$$

Then $A = B \cup \omega$ is the set of countable ordinals.

Fact 5. If $\beta < \alpha$ and α is countable, then β is countable.

The same arguments show the following.

Theorem 6. *For any cardinal κ, there is a cardinal $\lambda > \kappa$.*

Definition. For κ a cardinal, κ^+, the *cardinal successor* of κ, is the least cardinal $> \kappa$.

- $\omega_0 = \aleph_0 = \omega$.
- $\omega_1 = \aleph_1 = \omega_0^+$ is the least uncountable cardinal.

3 The Continuum Hypothesis

Definition. The *Continuum Hypothesis* (CH) is the statement

$$|\mathcal{P}(\omega_0)| = \omega_1.$$

Theorem 7 (Gödel, Cohen)**.** *If ZFC is consistent, then neither CH nor $\neg$CH is provable from ZFC.*

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Lecture 5 — March 15, 2023

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1 Cardinals

Definition (Cardinal)**.** An ordinal κ is a *cardinal* if and only if for all $\sigma < \kappa$, $\sigma \not\approx \kappa$, i.e. there is no bijection $f : \sigma \to \kappa$.

Lemma 3. *If κ is a cardinal and $\sigma < \kappa$, then there is no surjection $f : \sigma \to \kappa$.*

Proof. Fix an ordinal κ, some $\sigma < \kappa$, and a surjection $f : \sigma \to \kappa$. We show that κ is not a cardinal. Let

$$S = \{\alpha < \sigma \mid \forall \beta < \alpha \ f(\beta) \neq f(\alpha)\}.$$

Observation. $f \restriction S$ is a bijection from S onto κ.

- $f \restriction S$ has to be one-to-one: if $\beta < \alpha$ are both in S, then $f(\beta) \neq f(\alpha)$ by the definition of S.
- $f \restriction S$ is onto: take $\gamma < \kappa$. Then $\gamma \in \operatorname{ran}(f)$ because f is onto, so $\gamma = f(\beta)$ for some β. But then for α least such that $f(\alpha) = \gamma$, we have $\alpha \in S$.

But $(S, \in)$ is a well-order and $S \subseteq \sigma$, so $\operatorname{type}(S, \in) \leq \sigma$. Let $\theta = \operatorname{type}(S, \in) \leq \sigma$ and

$$\pi : (\theta, \in) \cong (S, \in)$$

(in particular, $\pi : \theta \to S$ is a bijection). Then

$$(f \restriction S) \circ \pi : \theta \to \kappa$$

is a bijection, i.e. $\theta \approx \kappa$ and $\theta \leq \sigma < \kappa$, so κ is not a cardinal. ∎

Lemma 4. *Let κ, λ be cardinals. Then*

$$\kappa \leq \lambda \iff \textit{there is an injection } f : \kappa \to \lambda.$$

Proof. ($\Rightarrow$) Suppose $\kappa \leq \lambda$. Then $\mathrm{id}_\kappa : \kappa \to \kappa \subseteq \lambda$ is an injection.

($\Leftarrow$) Suppose $f : \kappa \to \lambda$ is an injection. Let $S = \operatorname{ran}(f)$, so $f : \kappa \to S$ is a bijection. But $\operatorname{type}(S) \leq \lambda$, because $S \subseteq \lambda$. So

$$\kappa = |\operatorname{type}(S)| \leq \operatorname{type}(S) \leq \lambda,$$

where the equality holds since κ is a cardinal and $\kappa \approx S \approx \operatorname{type}(S)$. ∎

Theorem 8 (Cantor–Bernstein–Schröder)**.** *Let A, B be sets. If there are injections $f : A \to B$ and $g : B \to A$, then $A \approx B$.*

Proof. Our proof will use AC, but this is not necessary; see Exercise 4.14.

Let $\kappa = |A|$ and $\lambda = |B|$. (The fact that A is in bijection with an ordinal requires AC.) Let $f\colon A \to B$ be an injection. Then there is an injection $f^*\colon \kappa \to \lambda$. Why? There are bijections $\pi\colon \kappa \to A$ and $\sigma\colon B \to \lambda$, and then

$$\sigma \circ f \circ \pi\colon \kappa \to \lambda$$

is an injection. (We used AC to show that every set is in bijection with some ordinal.) So $\kappa \leq \lambda$ by Lemma 4.

There is also an injection $g^*\colon \lambda \to \kappa$, so also $\lambda \leq \kappa$, i.e. $\kappa = \lambda$. So $|A| = |B|$, hence $A \approx B$. ■

Remark 4. We use Greek letters that come earlier in the alphabet $(\alpha, \beta, \gamma, \dots)$ for ordinals, and later ones $(\kappa, \lambda, \dots)$ for cardinals.

Lemma 5. *If $\alpha \leq \beta$, then $|\alpha| \leq |\beta|$. In particular, if κ is a cardinal and $\kappa \leq \beta$, then $\kappa \leq |\beta|$.*

Proof. Since $\alpha \leq \beta$, $\mathrm{id}_\alpha\colon \alpha \to \alpha \subseteq \beta$ is an injection. So there is an injection from $|\alpha|$ into $|\beta|$. But then Lemma 4 gives $|\alpha| \leq |\beta|$. ■

Lemma 6. *If A is a set of cardinals, then* $\sup A$ *is a cardinal.*

Proof. Case 1: A has a maximum element κ. Then $\sup A = \kappa$, so $\sup A$ is a cardinal.

Case 2: A has no maximum element. Then $\sup A > \kappa$ for any $\kappa \in A$. For any $\kappa \in A$ (so κ is a cardinal), Lemma 5 gives

$$\kappa \leq \sup A \implies \kappa \leq |\sup A|.$$

But $\sup A$ is the least ordinal γ such that $\kappa \leq \gamma$ for all $\kappa \in A$. So $\sup A \leq |\sup A|$.

But $|\sup A|$ is a cardinal and $\sup A \approx |\sup A|$, so $|\sup A| \leq \sup A$ (for any ordinal α, $|\alpha| \leq \alpha$). So $\sup A = |\sup A|$, which is a cardinal. ■

2 The Alephs

Definition. Define cardinals $\omega_\alpha = \aleph_\alpha$, for α an ordinal, by recursion on α:

- $\aleph_0 = \omega$;
- $\aleph_{\alpha+1} = \aleph_\alpha^+$, the least cardinal $> \aleph_\alpha$;
- $\aleph_\lambda = \sup\{\aleph_\alpha \mid \alpha < \lambda\}$ if λ is a limit.

Definition. A *successor cardinal* is a cardinal of the form κ^+ (the least cardinal above κ) for some cardinal κ.

A *limit cardinal* is a cardinal which is not a successor cardinal and is $\neq 0$.

Theorem 9. *For every infinite cardinal κ, there is an α such that $\kappa = \aleph_\alpha$.*

21-329: Set Theory Spring 2023

Lecture 6 — March 17, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Successor and Limit Cardinals

Recall:

Definition.

- A cardinal κ is a *successor cardinal* if and only if $\kappa = \lambda^+$ for some cardinal λ.
- A cardinal κ is a *limit cardinal* if $\kappa \neq 0$ and κ is not a successor.

Basic observation. If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.

Theorem 10. *Let κ be an infinite cardinal. Then $\kappa = \aleph_\alpha$ for some α.*

Proof. By transfinite induction on the infinite cardinals.

Base case: $\kappa = \omega$ (the least infinite cardinal). Then $\kappa = \omega = \aleph_0$.

Successor step. Let $\kappa > \omega$ be a successor cardinal. Then $\kappa = \lambda^+$ for a cardinal $\lambda \geq \omega$. The induction hypothesis gives $\lambda = \aleph_\alpha$ for some α. But then

$$\kappa = \lambda^+ = \aleph_\alpha^+ = \aleph_{\alpha+1}.$$

Limit step. Let $\kappa > \omega$ be a limit cardinal. We claim

$$\kappa = \sup\{\lambda \in \kappa \mid \lambda \text{ is a cardinal and } \lambda \geq \omega\}.$$

This uses the following fact.

Fact 6 (cofinal sets)**.** Let $S \subseteq T$ be sets of ordinals. If for every $\alpha \in T$ there is a $\beta \in S$ with $\beta \geq \alpha$, then $\sup S = \sup T$.

Apply the fact to $T = \kappa$ and

$$S = \{\lambda \in \kappa \mid \lambda \geq \omega \text{ and } \lambda \text{ is a cardinal}\}.$$

Let $\alpha \in \kappa$. Then $|\alpha| \leq \alpha < |\alpha|^+$. So either $\alpha < \omega \in S$, or else $\alpha \geq \omega$ and $|\alpha|^+ \in S$.

Why? Since $\alpha \in \kappa$ and $|\alpha| \leq \alpha$, we have $|\alpha| \in \kappa$. Now $|\alpha|^+$ is the least cardinal $> |\alpha|$, and κ is a cardinal $> |\alpha|$, so $\kappa \geq |\alpha|^+$. But κ is not a successor cardinal, so $\kappa \neq |\alpha|^+$, i.e. $\kappa > |\alpha|^+$.

Let

$$\alpha = \sup\{\beta \mid \exists \lambda < \kappa \ \ \lambda = \aleph_\beta\}.$$

Since κ is a limit cardinal, this set has no largest element, so α is a limit ordinal and the set is exactly α. Then

$$\begin{aligned}\kappa &= \sup\{\lambda < \kappa \mid \lambda \text{ is an infinite cardinal}\} \\ &= \sup\{\aleph_\beta \mid \beta < \alpha\},\end{aligned}$$

where the second equality uses the induction hypothesis. So $\kappa = \aleph_\alpha$. ■

2 Cardinal Arithmetic

Definition. For cardinals κ, λ,

$$\begin{aligned}\kappa \oplus \lambda &\stackrel{\text{def}}{=} |\kappa + \lambda| = \big|(\{0\} \times \kappa) \cup (\{1\} \times \lambda)\big|, \\ \kappa \otimes \lambda &\stackrel{\text{def}}{=} |\kappa \cdot \lambda| = |\lambda \times \kappa|.\end{aligned}$$

Taking the cardinality forgets the order completely.

Observations.

- Cardinal addition and multiplication are commutative (unlike their ordinal counterparts).
- They agree with the usual addition and multiplication on the natural numbers.

Lemma 7. *Let λ be an infinite cardinal. Then $\lambda \otimes \lambda = \lambda$.*

Proof. By induction on the infinite cardinals $\lambda \geq \omega$. The induction hypothesis is that for all μ with $\omega \leq \mu < \lambda$, $\mu \otimes \mu = \mu$.

We'll produce a well-ordering $\lhd$ of $\lambda \times \lambda$ such that $\text{type}(\lambda \times \lambda, \lhd) = \lambda$. (So then there is an isomorphism $\pi\colon (\lambda, \in) \to (\lambda \times \lambda, \lhd)$, which is in particular a bijection from λ to $\lambda \times \lambda$.)

Recall that $<_{\text{lex}}$ on $\lambda \times \lambda$ is defined as follows: for $\alpha, \bar{\alpha}, \beta, \bar{\beta} \in \lambda$,

$$(\bar{\alpha}, \bar{\beta}) <_{\text{lex}} (\alpha, \beta) \iff \text{either } \bar{\alpha} < \alpha, \text{ or else } \bar{\alpha} = \alpha \text{ but } \bar{\beta} < \beta.$$

Now define $\lhd$ on $\lambda \times \lambda$ by $(\bar{\alpha}, \bar{\beta}) \lhd (\alpha, \beta)$ if and only if either

- $\max(\bar{\alpha}, \bar{\beta}) < \max(\alpha, \beta)$, or
- $\max(\bar{\alpha}, \bar{\beta}) = \max(\alpha, \beta)$ but $(\bar{\alpha}, \bar{\beta}) <_{\text{lex}} (\alpha, \beta)$.

How does this start? The least element is $(0, 0)$, and

$$(0,0) \lhd (0,1) \lhd (1,0) \lhd (1,1) \lhd (0,2) \lhd (1,2) \lhd \cdots$$

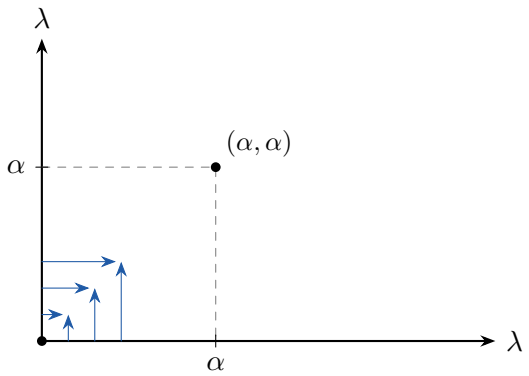

Figure 5: The order $\lhd$ on $\lambda \times \lambda$ (first coordinate horizontal): pairs are enumerated shell by shell, each shell being the pairs with a given maximum.

Next: the proof continues next lecture.

21-329: Set Theory Spring 2023

Lecture 7 — March 20, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Recap

Lemma 8. *Let λ be an infinite cardinal. Then $\lambda \otimes \lambda = \lambda$.*

Recall the well-ordering $\lhd$ of $\lambda \times \lambda$: $(\bar{\alpha}, \bar{\beta}) \lhd (\alpha, \beta)$ if and only if

- $\max(\bar{\alpha}, \bar{\beta}) < \max(\alpha, \beta)$, or
- $\max(\bar{\alpha}, \bar{\beta}) = \max(\alpha, \beta)$ and $(\bar{\alpha}, \bar{\beta}) <_{\text{lex}} (\alpha, \beta)$.

Graphically:

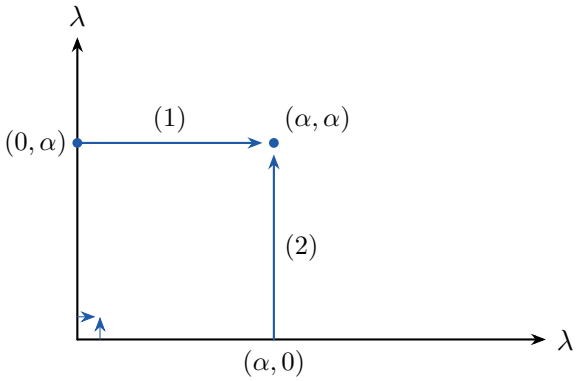

Figure 6: The pairs with maximum α come in $\lhd$-order along (1), then along (2); within (1) and (2) the order is lexicographic.

2 Proof of the Lemma

Claim 1. $\lhd$ is a well-ordering.

Proof. It is clearly a linear ordering. So let $S \subseteq \lambda \times \lambda$ with $S \neq \emptyset$. We just need to show that S has a $\lhd$-minimal element.

Let

$$\gamma = \min\{\max(\alpha, \beta) \mid (\alpha, \beta) \in S\}, \qquad S' = \{(\alpha, \beta) \in S \mid \max(\alpha, \beta) = \gamma\}.$$

For $(\alpha, \beta), (\bar{\alpha}, \bar{\beta}) \in S'$,

$$(\bar{\alpha}, \bar{\beta}) \lhd (\alpha, \beta) \iff (\bar{\alpha}, \bar{\beta}) <_{\text{lex}} (\alpha, \beta).$$

Since $<_{\text{lex}}$ is a well-ordering of $\lambda \times \lambda$, and by our choice of γ the set S' is nonempty, S' has a $<_{\text{lex}}$-minimal element (α, β). Then (α, β) is also the $\lhd$-minimal element of S. ■

Claim 2. $\text{type}(\lambda \times \lambda, \lhd) = \lambda$.

Proof. We'll show that for every $\alpha, \beta \in \lambda$,

$$\text{type}\big(\text{pred}_{\lhd}(\alpha, \beta), \lhd\big) < \lambda,$$

where

$$\text{pred}_{\lhd}(\alpha, \beta) = \{(\bar{\alpha}, \bar{\beta}) \in \lambda \times \lambda \mid (\bar{\alpha}, \bar{\beta}) \lhd (\alpha, \beta)\}$$

is the set of predecessors of (α, β).

We'll do this by induction. So assume that for all infinite cardinals μ with $\omega \leq \mu < \lambda$, $\text{type}(\mu \times \mu, \lhd) = \mu$.

Base case: $\lambda = \omega$. For any $n, m \in \omega$, $\text{pred}_{\lhd}(n, m)$ is a finite set.

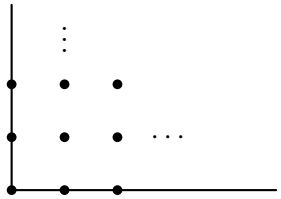

Figure 7: The points of $\omega \times \omega$: each has only finitely many $\lhd$-predecessors.

So $\text{type}(\text{pred}_{\lhd}(n, m)) < \omega$. Since $(\omega \times \omega, \lhd)$ has no maximal element, the sup of these types is ω, so $\text{type}(\omega \times \omega, \lhd) = \omega$.

Inductive step: $\lambda \geq \omega_1$. Fix (α, β). Let

$$\gamma = \max(\alpha + 1, \beta + 1, \omega) \quad \text{and} \quad \mu = |\gamma|.$$

- $\mu \geq \omega$ by the choice of γ.
- $\mu < \lambda$ because $\omega, \alpha, \beta < \lambda$ and λ is a cardinal.

We want

$$\text{type}\big(\text{pred}_{\lhd}(\alpha, \beta), \lhd\big) < \mu^+ \leq \lambda.$$

We have $\text{pred}_{\lhd}(\alpha, \beta) \subseteq \gamma \times \gamma$. We know $|\gamma| = \mu$, so $\gamma \approx \mu$, and so $\gamma \times \gamma \approx \mu \times \mu$. The induction hypothesis gives $|\mu \times \mu| = \mu$. So, letting $\rho = \text{type}(\gamma \times \gamma, \lhd)$, we get $|\rho| = \mu$. So $\rho < \mu^+$. But

$$\text{type}\big(\text{pred}_{\lhd}(\alpha, \beta), \lhd\big) \leq \text{type}(\gamma \times \gamma, \lhd),$$

so $\text{type}(\text{pred}_{\lhd}(\alpha, \beta), \lhd) < \mu^+$.

So $\text{type}(\lambda \times \lambda, \lhd) = \lambda$:

- it is $\leq \lambda$ because $\text{type}(\text{pred}_{\lhd}(\alpha, \beta), \lhd) < \lambda$ for all $\alpha, \beta \in \lambda$;
- it is $\geq \lambda$ because there is an injection $\lambda \hookrightarrow \lambda \times \lambda$ which preserves order. ■

3 Sums and Products of Infinite Cardinals

Theorem 11. *For any infinite cardinals κ, λ,*

$$\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda).$$

Proof. Without loss of generality $\kappa \leq \lambda$. We show $\kappa \oplus \lambda = \kappa \otimes \lambda = \lambda$:

$$\begin{aligned} \lambda \leq \kappa \oplus \lambda &\leq \lambda \oplus \lambda \\ &= 2 \otimes \lambda \\ &\leq \kappa \otimes \lambda \\ &\leq \lambda \otimes \lambda \\ &= \lambda. \end{aligned}$$

Here $\lambda \oplus \lambda = 2 \otimes \lambda$ because $\lambda + \lambda = \lambda \cdot 2$ and $|\lambda \cdot 2| = |2 \cdot \lambda| = 2 \otimes \lambda$. ■

Remark 5. On the homework, feel free to use

- $\omega \otimes \omega = \omega$;
- countable unions of countable sets are countable.

21-329: Set Theory Spring 2023

Lecture 8 — March 22, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Announcements

- Midterm on Friday, 3/31.
- Next Monday's and Wednesday's lectures are via Zoom.

2 Countable Unions

Last time:

Theorem 12. *$\lambda \otimes \lambda = \lambda$ for any infinite cardinal λ.*

In particular, $|\omega \times \omega| = \omega$.

Lemma 9. *A countable union of countable sets is countable. (That is, if $\mathcal{F}$ is countable and every $A \in \mathcal{F}$ is countable, then $\bigcup \mathcal{F}$ is countable.)*

Proof. The difficult case is when $|\mathcal{F}| = \omega$ and $|A| = \omega$ for all $A \in \mathcal{F}$. So for every $A \in \mathcal{F}$ there is a bijection $f : \omega \to A$.

Without loss of generality we may assume that $A \cap B = \emptyset$ for all distinct $A, B \in \mathcal{F}$. Why? Since $|\mathcal{F}| = \omega$, fix a bijection $g : \omega \to \mathcal{F}$, and write $\langle A_n \mid n \in \omega \rangle$ for $A_n = g(n)$. We'll produce a new sequence $\langle A'_n \mid n \in \omega \rangle$ of pairwise disjoint sets such that

$$\bigcup \mathcal{F} = \bigcup \{A_n \mid n \in \omega\} = \bigcup \{A'_n \mid n \in \omega\}.$$

Define A'_n by recursion:

$$\begin{aligned}
A'_0 &= A_0, \\
A'_1 &= A_1 - A_0 = A_1 - A'_0, \\
A'_2 &= A_2 - (A_0 \cup A_1) = A_2 - (A'_0 \cup A'_1), \\
&\vdots \\
A'_{n+1} &= A_{n+1} - (A_0 \cup \cdots \cup A_n) = A_{n+1} - (A'_0 \cup \cdots \cup A'_n),
\end{aligned}$$

where the second equality in each line uses the induction hypothesis.

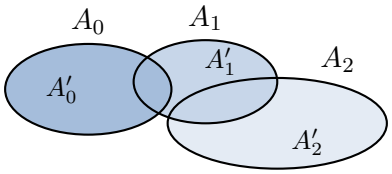

Figure 8: Making the sets disjoint: A'_n is the part of A_n not already covered by $A_0, \ldots, A_{n-1}$.

Let $g\colon \omega \to \mathcal{F}$ be a bijection. By AC (this needs a very weak form of choice, the principle of dependent choice, but it is still choice), there is $\langle f_n \mid n \in \omega \rangle$ such that for all n, $f_n\colon \omega \to A_n$ is a bijection.

Define $\pi\colon \omega \times \omega \to \bigcup \mathcal{F}$ by

$$\pi(n, m) = f_n(m).$$

Check that π is a bijection. But $|\omega \times \omega| = \omega$, so there is a bijection $\sigma\colon \omega \to \omega \times \omega$. But then $\pi \circ \sigma\colon \omega \to \bigcup \mathcal{F}$ is a bijection. So $|\bigcup \mathcal{F}| = \omega$. ■

Feel free to use this lemma on the homework.

Lemma 10. *If $S \subseteq \omega_1$ and S is countable, then S is bounded in ω_1, i.e. there is $\alpha < \omega_1$ such that $S \subseteq \alpha$ (which implies $\sup S < \omega_1$).*

Feel free to use this lemma on the homework too.

Proof. Fix a countable $S \subseteq \omega_1$. It suffices to show $\sup S < \omega_1$ (then $\sup S$ is our desired α).

By definition $\sup S = \bigcup S$, and every $\alpha \in \omega_1$ is countable (since ω_1 is the set of all countable ordinals). So $\bigcup S$ is a countable union of countable sets ($\alpha \in S \implies \alpha \in \omega_1$, so α is countable).

So $\sup S = \bigcup S$ is countable, so $\sup S$ is a countable ordinal, i.e. $\sup S < \omega_1$. ■

3 Cardinal Exponentiation

Definition (Functions from A to B)**.** For sets A, B,

$$^A B = \{f \mid f\colon A \to B\}.$$

Definition. For cardinals κ, λ, let

$$\lambda^\kappa = |^\kappa \lambda|.$$

Warning: this is *not* ordinal exponentiation.

This definition is highly nontrivial.

Observation. n^m is the usual exponentiation for $n, m \in \omega$. But this is not true in general.

Lemma 11. $|\mathcal{P}(A)| = 2^{|A|}$. *(This holds for infinite and finite cardinals!)*

Proof. We have $2^{|A|} = |{}^A 2|$. So it suffices to produce a bijection

$$c\colon \mathcal{P}(A) \to {}^A 2 = {}^A\{0,1\},$$

the set of functions from A to $\{0,1\}$ (c sends a subset of A to its characteristic function).

For $X \subseteq A$, let $c_X\colon A \to \{0,1\}$ be given by

$$c_X(a) = \begin{cases} 0 & \text{if } a \notin X, \\ 1 & \text{if } a \in X. \end{cases}$$

Then put $c(X) = c_X$. Check that c is a bijection. ■

Corollary 1. *$2^\kappa \geq \kappa^+$ for any cardinal κ.*

Proof. Cantor's theorem told us $|\mathcal{P}(\kappa)| \geq \kappa^+$. But $2^\kappa = |\mathcal{P}(\kappa)|$. ■

Next time: we will prove the following.

Lemma 12. *Let κ, λ, μ be cardinals. Then $(\mu^\lambda)^\kappa = \mu^{\lambda \otimes \kappa}$.*

4 Cofinality

This begins Section 4.3.

Definition (Cofinality)**.** For λ a limit ordinal, the *cofinality* of λ, $\mathrm{cf}(\lambda)$, is the least ordinal κ such that there is $f\colon \kappa \to \lambda$ with $\mathrm{ran}(f)$ *cofinal* in λ, i.e. for all $\alpha \in \lambda$ there is $\beta \in \mathrm{ran}(f)$ with $\beta \geq \alpha$. (Equivalently, $\sup(\mathrm{ran}(f)) = \lambda$.)

The idea: the range is unbounded, so it has the full supremum.

Observation. $\mathrm{cf}(\lambda) \leq |\lambda| \leq \lambda$.

Example 3.

- $\mathrm{cf}(\omega) = \omega$. Why? If $f\colon n \to \omega$, then $\mathrm{ran}(f)$ is finite, hence bounded in ω.
- $\mathrm{cf}(\omega + \omega) = \omega$: take $f\colon \omega \to \omega + \omega$ given by $f(n) = \omega + n$. Its range $\{\omega + n \mid n \in \omega\}$ is cofinal.
- $\mathrm{cf}(\omega_1) = \omega_1$: if $f\colon \omega \to \omega_1$, then $\mathrm{ran}(f)$ is countable, so $\mathrm{ran}(f)$ is bounded, i.e. $\sup \mathrm{ran}(f) < \omega_1$.

21-329: Set Theory Spring 2023

Lecture 9 — March 24, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Exponentiation of Exponents

Lemma 13. *For cardinals λ, μ, κ,*

$$(\mu^\lambda)^\kappa = \mu^{\lambda \otimes \kappa}.$$

Proof. Note that

$$(\mu^\lambda)^\kappa = \left|{}^\kappa({}^\lambda\mu)\right| \qquad \text{and} \qquad \mu^{\lambda\otimes\kappa} = \left|{}^{\kappa\times\lambda}\mu\right|.$$

We'll exhibit a bijection

$$F\colon {}^\kappa({}^\lambda\mu) \to {}^{\kappa\times\lambda}\mu.$$

For $g \in {}^\kappa({}^\lambda\mu)$ and $\alpha < \kappa$, we have $g(\alpha) \in {}^\lambda\mu$, i.e. $g(\alpha)\colon \lambda \to \mu$. For $\beta < \lambda$, $g(\alpha)(\beta)$ is an ordinal $< \mu$.

Let $F(g)$ be the function in ${}^{\kappa\times\lambda}\mu$ such that for all $\alpha < \kappa$ and $\beta < \lambda$,

$$F(g)(\alpha, \beta) = g(\alpha)(\beta).$$ ■

2 Cofinality

Definition (Cofinal function)**.** Let κ, λ be limit ordinals and $f\colon \kappa \to \lambda$. Then f is *cofinal* if and only if $\sup f[\kappa] = \lambda$ (equivalently, $f[\kappa]$ is unbounded, also called cofinal), i.e. for all $\alpha < \lambda$ there is $\beta \in f[\kappa]$ with $\beta \geq \alpha$.

Definition (Cofinality)**.** For λ a limit ordinal, the *cofinality* of λ, $\mathrm{cf}(\lambda)$, is the least ordinal κ such that there is $f\colon \kappa \to \lambda$ which is cofinal.

Definition (Singular and regular)**.**

- λ is *singular* if $\mathrm{cf}(\lambda) < \lambda$.
- λ is *regular* if $\mathrm{cf}(\lambda) = \lambda$.

Observation. $\mathrm{cf}(\lambda) \leq |\lambda| \leq \lambda$.

Proof. Let $\kappa = |\lambda|$. Then there is a bijection $f\colon \kappa \to \lambda$. Clearly, then, $\sup f[\kappa] = \lambda$. ■

Example 4.

- $\mathrm{cf}(\omega) = \omega$, so ω is regular.

- $\mathrm{cf}(\omega + \omega) = \omega$.
- $\mathrm{cf}(\omega_1 + \omega) = \omega$, using $f(n) = \omega_1 + n$.
- $\mathrm{cf}(\omega_1) = \omega_1$, so ω_1 is regular. Countable subsets $S \subseteq \omega_1$ are bounded, so unbounded subsets $S \subseteq \omega_1$ have to be uncountable, of size ω_1.

The second and third are singular; for these, $\mathrm{cf}(\lambda) < |\lambda|$. We don't care about successor ordinals when talking about cofinality.

Theorem 13. *For any cardinal κ, κ^+ is regular (i.e. successor cardinals are regular). So $\mathrm{cf}(\kappa^+) = \kappa^+$.*

Theorem 14. *For any limit ordinal λ, $\mathrm{cf}(\lambda)$ is a regular cardinal.*

Lemma 14. *For any limit ordinal λ, there is $g\colon \mathrm{cf}(\lambda) \to \lambda$ which is strictly increasing and cofinal (i.e. for all $\alpha, \beta < \mathrm{cf}(\lambda)$, $\alpha < \beta \implies g(\alpha) < g(\beta)$).*

Proof. Let $\kappa = \mathrm{cf}(\lambda)$ and fix a cofinal $f\colon \kappa \to \lambda$. We define $g\colon \kappa \to \lambda$, cofinal and strictly increasing, by transfinite recursion (i.e. recursion on κ).

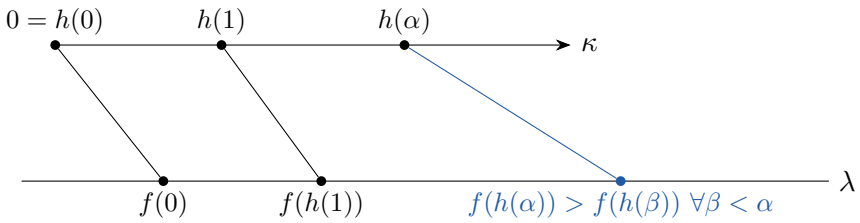

Figure 9: Choosing $h(\alpha) < \kappa$ so that $f(h(\alpha))$ lies above all earlier values $f(h(\beta))$.

Define h by transfinite recursion: $h(0) = 0$, and for $\alpha < \kappa$,

$$h(\alpha) = \text{the least ordinal } \xi \text{ such that } f(\xi) > f(h(\beta)) \text{ for all } \beta < \alpha,$$

if there is such a ξ. Stop otherwise.

Then h has domain some ordinal $\gamma \leq \kappa$, and $h\colon \gamma \to \kappa$.

Claim 3. $\gamma = \kappa$.

Proof. By choice of h, $f \circ h\colon \gamma \to \lambda$ is cofinal (otherwise, we could keep defining h). But then $\gamma < \kappa$ is impossible, because $\kappa = \mathrm{cf}(\lambda)$. ■

Let $g = f \circ h$. Then $g\colon \kappa \to \lambda$ is cofinal. But by choice of h, g is strictly increasing. ■

21-329: Set Theory Spring 2023

Lecture 10 — March 27, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Announcements

Midterm 2 is on Friday. It is similar to the homework. Key topics:

- ordinal arithmetic;
- basics of cardinals;
- cardinal arithmetic (not including today);
- basics of cofinality (feel free to use that $\mathrm{cf}(\lambda)$ is regular, but you shouldn't need to use it).

2 Cofinality

Recall: for λ a limit ordinal, $\mathrm{cf}(\lambda)$ is the least κ such that there is a cofinal $f\colon \kappa \to \lambda$, i.e. $\sup f[\kappa] = \lambda$, or equivalently, for all $\alpha < \lambda$ there is $\beta \in f[\kappa]$ such that $\alpha \leq \beta$.

λ is *regular* if $\mathrm{cf}(\lambda) = \lambda$ (e.g. ω_1, ω), and *singular* otherwise (i.e. $\mathrm{cf}(\lambda) < \lambda$; e.g. $\omega + \omega$ and $\aleph_\omega$, both of cofinality ω).

Theorem 15. *For all λ, $\mathrm{cf}(\lambda)$ is a regular cardinal.*

Lemma 15. *For all λ (implicitly limit), there is $g\colon \mathrm{cf}(\lambda) \to \lambda$ which is cofinal and strictly increasing.*

Homework 3.17. For all limit κ, λ, μ, if $f\colon \kappa \to \lambda$ and $g\colon \lambda \to \mu$ are such that f, g are cofinal and g is nondecreasing, then $g \circ f\colon \kappa \to \mu$ is cofinal.

Proof of Theorem 15. We use the two things above. Fix μ limit, and let $\lambda = \mathrm{cf}(\mu)$. We want to show that λ is a regular cardinal.

Notice that if $|\lambda| < \lambda$, there is a surjection $f\colon |\lambda| \to \lambda$. Such an f is cofinal, so if λ is not a cardinal, then λ is not regular. So we want to show $\mathrm{cf}(\lambda) = \lambda$ (i.e. λ is regular).

Suppose not. Let $\kappa = \mathrm{cf}(\lambda) < \lambda$, and let $f\colon \kappa \to \lambda$ be cofinal. Apply Lemma 15 to μ to get a nondecreasing cofinal $g\colon \lambda \to \mu$. Apply Homework 3.17: $g \circ f\colon \kappa \to \mu$ is cofinal. But $\kappa = \mathrm{cf}(\lambda) < \lambda = \mathrm{cf}(\mu)$, a contradiction. ∎

Fact 7. ω_1 is regular (just use that countable unions of countable sets are countable).

Theorem 16. *Successor cardinals are regular.*

Proof. Let κ be a cardinal. We want to show that κ^+ is regular.

Let $f\colon \eta \to \kappa^+$ with $\eta < \kappa^+$. We'll show $\sup f[\eta] < \kappa^+$, which will follow from showing $|f[\eta]| < \kappa^+$.

(Equivalent fact: a union of fewer than κ^+ sets, each of size $< \kappa^+$, has size $< \kappa^+$.)

It suffices to show $|\sup f[\eta]| \leq \kappa$. We have

$$\begin{aligned} |\sup f[\eta]| \leq \Big|\bigcup_{\alpha<\eta} f(\alpha)\Big| &\leq |\eta| \otimes \sup_{\alpha<\eta} |f(\alpha)| \qquad (*) \\ &\leq \kappa \otimes \kappa \qquad \text{(since each is } < \kappa^+) \\ &= \kappa. \end{aligned}$$

Why $(*)$ is true: it is enough to get a surjection from $\kappa \otimes \kappa$ onto $\bigcup_{\alpha<\eta} f(\alpha)$. Each $f(\alpha)$ has cardinality $\leq \kappa$, so there is a surjection $g_\alpha\colon \kappa \to f(\alpha)$. Since $|\eta| \leq \kappa$, there is a surjection $h\colon \kappa \to \eta$. Let $\pi\colon \kappa \times \kappa \to \bigcup_{\alpha<\eta} f(\alpha)$ be the surjection given by

$$\pi(\alpha, \beta) = g_{h(\alpha)}(\beta).$$

■

3 König's Theorem

Theorem 17 (König)**.** *If λ is an infinite cardinal, then $\lambda^{\mathrm{cf}(\lambda)} > \lambda$.*

Proof. Let $\kappa = \mathrm{cf}(\lambda)$, and fix a cofinal $f\colon \kappa \to \lambda$. We have $|{}^\kappa\lambda| = \lambda^\kappa$, so it is enough to show there is no surjection $G\colon \lambda \to {}^\kappa\lambda$.

Fix some $G\colon \lambda \to {}^\kappa\lambda$. We want to find $h \notin \mathrm{ran}(G)$. For $\alpha < \kappa$, let

$$A_\alpha = \{G(\eta)(\alpha) \mid \eta < f(\alpha)\}.$$

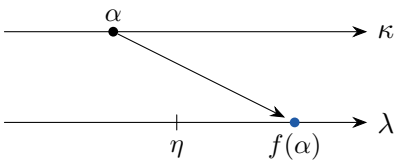

Figure 10: A_α collects the values $G(\eta)(\alpha)$ for the η below $f(\alpha)$.

Note $A_\alpha \subseteq \lambda$ and $|A_\alpha| \leq |f(\alpha)| < \lambda$. In particular $A_\alpha \neq \lambda$ (as λ is a cardinal). So let

$$h(\alpha) = \text{the least element of } \lambda - A_\alpha.$$

We want to show $h \neq G(\eta)$ for any $\eta < \lambda$. Why? Since f is cofinal, $\eta < f(\alpha)$ for some α. But then $G(\eta)(\alpha) \in A_\alpha$, so $h(\alpha) \neq G(\eta)(\alpha)$ because $h(\alpha) \notin A_\alpha$. So $h \neq G(\eta)$, because h and $G(\eta)$ disagree at input α.

So G is not surjective. ■

Corollary 2 (König)**.** *If κ is an infinite cardinal, then* $\operatorname{cf}(2^\kappa) > \kappa$.

Proof. We have $(2^\kappa)^{\operatorname{cf}(2^\kappa)} > 2^\kappa$ by König's theorem. But if $\mu \le \kappa$, then

$$(2^\kappa)^\mu = 2^{\kappa \otimes \mu} = 2^\kappa,$$

since $\kappa \otimes \mu = \max(\mu, \kappa) = \kappa$. So $\operatorname{cf}(2^\kappa) \not\le \kappa$, i.e. $\kappa < \operatorname{cf}(2^\kappa)$. ■

In particular, $\operatorname{cf}(2^\omega) > \omega$.

Theorem 18 (Gödel–Cohen)**.** *Our axioms ZFC can't prove $2^\omega = \omega_1$ or $2^\omega \ne \omega_1$. (A Fields Medal was awarded for this result.)*

But König's result implies $2^\omega \ne \aleph_\omega$ (since $\operatorname{cf}(\aleph_\omega) = \omega$).

21-329: Set Theory Spring 2023

Lecture 11 — March 29, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Problem 1

Part (a)

Let $f\colon \omega_1 \to \omega_1$ and $\xi < \omega_1$. Show there is $\beta < \omega_1$ such that $\xi \leq \beta$ and

$$\forall \alpha < \beta \quad f(\alpha) < \beta,$$

i.e. β is closed under f: $f \restriction \beta\colon \beta \to \beta$.

Proof. Define ordinals $\alpha_n < \omega_1$, for $n < \omega$, by recursion on ω as follows (or a function $g\colon \omega \to \omega_1$, $g(n) = \alpha_n$). Let

$$\alpha_0 = \xi,$$
$$\alpha_{n+1} = \sup f[\alpha_n + 1].$$

(Roughly, $\alpha_n = f \circ \cdots \circ f(\xi)$, with n applications of f.) Let

$$\beta = \sup\{\alpha_n \mid n \in \omega\} = \bigcup\{\alpha_n \mid n \in \omega\}.$$

Then

$$\alpha < \beta \implies \alpha \leq \alpha_n \text{ for some } n \implies f(\alpha) < \alpha_{n+1} < \beta.$$

To show the α_n are well defined, note that for any $\gamma < \omega_1$, $\sup f[\gamma] < \omega_1$ because ω_1 is regular: $f[\gamma]$ is countable, and any countable subset of ω_1 is bounded in ω_1.

Also $\beta < \omega_1$, as $\{\alpha_n\}_{n\in\omega}$ is also a countable subset of ω_1. ■

Part (b)

Suppose $g\colon \omega_1 \to \omega_1$ is increasing (i.e. $\alpha < \beta \implies g(\alpha) < g(\beta)$) and continuous at limit ordinals, i.e. $g(\lambda) = \sup g[\lambda]$ for all limit λ. Show that for all $\xi < \omega_1$ there is $\beta < \omega_1$ with $\beta > \xi$ and $g(\beta) = \beta$.

Proof. Apply part (a) to g to get $\beta < \omega_1$ with $\xi \leq \beta$ such that $g(\alpha) < \beta$ for all $\alpha < \beta$.

But g is increasing, so the order type $\mathrm{otp}(g[\beta], <)$ is β. So

$$g[\beta] \subseteq \beta \quad \text{and} \quad \mathrm{otp}(g[\beta], <) = \beta,$$

so $\sup g[\beta] = \beta$ (since there is an injective map from β into $\sup g[\beta]$ which is strictly increasing).

But since g is increasing, the β produced by part (a) is a limit ordinal (the α_n are strictly increasing: $n < m \implies \alpha_n < \alpha_m$). So

$$g(\beta) = \sup g[\beta] = \beta.$$ ■

2 Problem 2

Show $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.

Proof. It is easy to show $2^{\aleph_0} \leq \aleph_0^{\aleph_0}$, since ${}^{\omega}2 \subseteq {}^{\omega}\omega$.

Recall

$$\aleph_0 \otimes \aleph_0 = \aleph_0 \qquad \text{and} \qquad 2^{\aleph_0 \otimes \aleph_0} = 2^{\aleph_0},$$

so $|{}^{\omega \times \omega}2| = 2^{\aleph_0}$. So it suffices to get an injection

$$F \colon {}^{\omega}\omega \to {}^{\omega \times \omega}2.$$

For $f \colon \omega \to \omega$, let

$$F(f)(n,m) = \begin{cases} 1 & \text{if } f(n) = m, \\ 0 & \text{otherwise } (f(n) \neq m). \end{cases}$$

This is an injection: let $f, g \colon \omega \to \omega$ with $f \neq g$. Then there is n with $f(n) \neq g(n)$, and

$$F(f)(n, f(n)) = 1,$$
$$F(g)(n, f(n)) = 0$$

(because $F(g)(n,m) = 0$ whenever $g(n) \neq m$). So $F(f) \neq F(g)$. ∎

3 Problem 3

Let κ, λ be limit ordinals, and let $f \colon \kappa \to \lambda$ be cofinal and increasing, i.e. for all $\alpha < \lambda$ there is $\beta \in \kappa$ with $f(\beta) \geq \alpha$. Show $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Proof. Let $\mu = \mathrm{cf}(\kappa)$ and $\delta = \mathrm{cf}(\lambda)$. We want to show $\delta = \mu$.

Claim 4. $\delta \leq \mu$.

Proof. Let $g \colon \mu \to \kappa$ be cofinal. Then by Homework 3.17 (important for the exam), $f \circ g \colon \mu \to \lambda$ is cofinal. So $\mu \geq \mathrm{cf}(\lambda) = \delta$. ∎

Claim 5. $\mu \leq \delta$.

Fact. For all limit λ, there is $g \colon \mathrm{cf}(\lambda) \to \lambda$ which is cofinal and increasing.

So by the fact, let $g \colon \delta \to \lambda$ be cofinal and increasing, and let $h \colon \mu \to \kappa$ be increasing and cofinal. Then $f \circ h \colon \mu \to \lambda$ is increasing and cofinal.

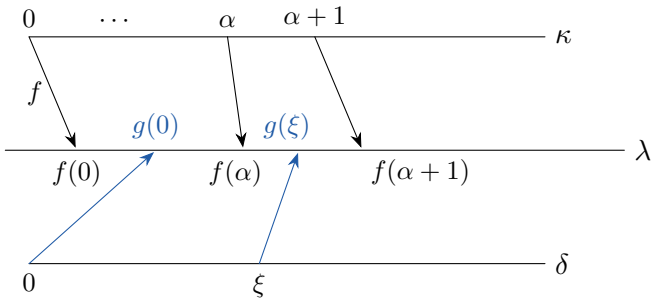

Figure 11: The cofinal maps $f\colon \kappa \to \lambda$ and $g\colon \delta \to \lambda$; here $g(\xi)$ falls between $f(\alpha)$ and $f(\alpha+1)$.

We want to define an increasing map $\pi\colon \delta \to \mu$. Let

$$\pi(\xi) = \text{the least } \gamma \text{ such that } f \circ h(\gamma) \geq g(\xi).$$

Then show that $\pi\colon \delta \to \mu$ and that π is increasing.

21-329: Set Theory Spring 2023

Lecture 12 — April 3, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Announcements

- No homework this week.
- We are starting Chapter 7 (later we'll come back to Chapter 5).

2 Review

Before Midterm 2 we showed:

Theorem 19 (König)**.** $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$.

Recall: we can't prove using our axioms ZFC whether or not $2^{\aleph_0} = \omega_1$. It turns out it is possible to have $2^{\aleph_0} = \omega_1$, or $2^{\aleph_0} = \omega_2$, ..., but König's theorem tells us $2^{\aleph_0} = \aleph_\omega$ is *not* possible.

Theorem 20. *Successor cardinals are regular.*

Question. Are there regular limit cardinals? This is independent from ZFC. Such cardinals are called *inaccessible cardinals*; they are examples of "large cardinals".

It turns out that from ZFC + "there is a regular limit cardinal" you can prove "ZFC is consistent".

Theorem 21 (Gödel's Second Incompleteness Theorem)**.** *For reasonable consistent theories T (like ZFC), T can't prove "T is consistent".*

3 Filters and Ideals

This is Chapter 7. Roughly, a filter will be a family $F \subseteq \mathcal{P}(X)$ capturing the idea that members of F are "large".

Definition (Filter)**.** Let $X \neq \emptyset$. A family $F \subseteq \mathcal{P}(X)$ is a *filter* over X if and only if:

(1) $\emptyset \notin F$;

(2) $X \in F$;

(3) for all $A, B \in \mathcal{P}(X)$, if $A \in F$ and $B \supseteq A$, then $B \in F$;

(4) for all $A, B \in F$, $A \cap B \in F$.

(Filters capture the idea of large sets.)

Example 5. Let F over ω be given by $A \in F \iff n \in A$, where n is a fixed natural number.

This is an example of a principal filter. A filter F over X is *principal* if there is $a \in X$ such that $A \in F \iff a \in A$.

Example 6 (Fréchet filter)**.** The filter over ω given by $A \in F \iff \omega - A$ is finite. This filter is *nonprincipal*, i.e. not principal.

Example 7 (Club filter on ω_1)**.** "Club" stands for *closed and unbounded*.

Definition (Unbounded)**.** A set $A \subseteq \omega_1$ is *unbounded* if and only if for all $\alpha \in \omega_1$ there is $\beta \in A$ with $\beta \geq \alpha$.

Definition (Closed)**.** A set $X \subseteq \omega_1$ is *closed* (in ω_1) if for any limit $\alpha < \omega_1$ such that $X \cap \alpha$ is unbounded in α, we have $\alpha \in X$.

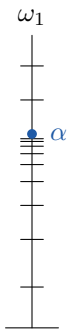

Figure 12: A closed set X (ticks): its elements below the limit α are unbounded in α, so $\alpha \in X$.

Definition (Club filter)**.** Let F over ω_1 be given by $A \in F \iff A$ is closed unbounded in ω_1.

Next time: F is a filter (in fact, a countably complete filter).

Definition (Ultrafilter)**.** A filter F over X is an *ultrafilter* if and only if for any $A \subseteq X$, either $A \in F$ or $X - A \in F$.

Observation.

- Principal filters are ultrafilters.
- The Fréchet filter is not an ultrafilter (e.g. consider the set of even numbers).
- The club filter is also not an ultrafilter.

Theorem 22 (Tarski)**.** *Every filter extends to an ultrafilter. That is, for any filter F over X, there is an ultrafilter G over X such that $F \subseteq G$.*

Corollary 3. *There are non-principal ultrafilters on ω.*

Exercise. Show that any ultrafilter extending the Fréchet filter is non-principal.

Informal exercise. Show that if G is an ultrafilter over X, there is no filter F over X such that $F \supsetneq G$.

Proof of Theorem 22. Let F be a filter over X. By AC, fix $\kappa = |\mathcal{P}(X)|$ and a bijection $f\colon \kappa \to \mathcal{P}(X)$. Let $\langle A_\alpha \mid \alpha < \kappa\rangle$ be given by $A_\alpha = f(\alpha)$. Note $\mathcal{P}(X) = \{A_\alpha \mid \alpha < \kappa\}$ because f is a surjection.

By recursion we'll build a sequence $\langle G_\alpha \mid \alpha < \kappa\rangle$ such that for all $\beta < \kappa$:

- G_β is a filter over X;
- for all $\alpha < \beta$, either $A_\alpha \in G_\beta$ or $X - A_\alpha \in G_\beta$;
- for all $\alpha < \beta$, $F \subseteq G_\alpha \subseteq G_\beta$.

Base step ($\beta = 0$). Set $G_0 = F$.

Successor step ($\beta = \alpha + 1$). Suppose we are given G_α, a filter as above. We'll define $G_{\alpha+1}$ ensuring either $A_\alpha \in G_{\alpha+1}$ or $X - A_\alpha \in G_{\alpha+1}$.

Case 1: for all $B \in G_\alpha$, $B \cap A_\alpha \neq \emptyset$. For $C \subseteq X$, put

$$C \in G_{\alpha+1} \iff \exists B \in G_\alpha \;\; B \cap A_\alpha \subseteq C.$$

We check that $G_{\alpha+1}$ is a filter extending G_α and containing A_α (as a member).

- $A_\alpha \in G_{\alpha+1}$: $X \in G_\alpha$ since G_α is a filter, so $A_\alpha \supseteq X \cap A_\alpha$ implies $A_\alpha \in G_{\alpha+1}$.
- $G_\alpha \subseteq G_{\alpha+1}$: let $B \in G_\alpha$. Trivially $B \supseteq B \cap A_\alpha$, so $B \in G_{\alpha+1}$.
- $G_{\alpha+1}$ is a filter. (1) $\emptyset \notin G_{\alpha+1}$ is the only non-trivial property, and it is immediate from the case hypothesis. The others follow from the definition: $X \in G_{\alpha+1}$ since $X \supseteq X \cap A_\alpha$; supersets of members are members; and if $B \cap A_\alpha \subseteq C$ and $B' \cap A_\alpha \subseteq C'$ with $B, B' \in G_\alpha$, then $(B \cap B') \cap A_\alpha \subseteq C \cap C'$ with $B \cap B' \in G_\alpha$.

Case 2: there is $B \in G_\alpha$ such that $B \cap A_\alpha = \emptyset$. Then we want to do the same thing, replacing A_α with $X - A_\alpha$.

Observation. For any filter, either $\forall B \in G_\alpha\ (B \cap A_\alpha \neq \emptyset)$ or $\forall B \in G_\alpha\ (B \cap (X - A_\alpha) \neq \emptyset)$.

Proposition 4. For any sequence $\langle G_\alpha \mid \alpha < \kappa\rangle$ of filters such that $G_\alpha \subseteq G_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\kappa} G_\alpha$ is a filter.

21-329: Set Theory Spring 2023

Lecture 13 — April 5, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Tarski's Theorem (continued)

Theorem 23 (Tarski)**.** *Every filter extends to an ultrafilter.*

Proof outline. Let F be a filter over X. First enumerate $\mathcal{P}(X)$ as $\langle A_\alpha \mid \alpha < \kappa \rangle$.

Build $\langle G_\alpha \mid \alpha < \kappa \rangle$ by recursion such that for all $\beta < \kappa$:

- G_β is a filter;
- $F \subseteq G_\alpha \subseteq G_\beta$ for all $\alpha < \beta$;
- $A_\alpha \in G_\beta$ or $X - A_\alpha \in G_\beta$ for all $\alpha < \beta$.

Key observation (this is what the successor step gives: from G_α and A_α we get $G_{\alpha+1}$). For G a filter over X and $A \subseteq X$, if $A \cap B \neq \emptyset$ for all $B \in G$, then there is a filter $G' \supseteq G$ with $A \in G'$.

Lemma 16. *For G a filter over X and $A \subseteq X$, either*

(1) $\forall B \in G \;\; A \cap B \neq \emptyset$*, or*

(2) $\forall B \in G \;\; (X - A) \cap B \neq \emptyset$.

Proof of Lemma 16. Suppose not, i.e. $\neg(1)$ and $\neg(2)$. Let $B, C \in G$ be such that

$$A \cap B = \emptyset, \qquad (X - A) \cap C = \emptyset.$$

Since G is a filter, $B \cap C \in G$. But then

$$A \cap (B \cap C) = \emptyset \quad \text{and} \quad (X - A) \cap (B \cap C) = \emptyset,$$

so $B \cap C = \emptyset$. So $\emptyset \in G$, a contradiction. ■

Hence the successor case is done. We are now left with the limit case.

Lemma 17. *Let $\mathcal{S}$ be a set of filters over X such that for all $F, G \in \mathcal{S}$ there is $H \in \mathcal{S}$ with $F, G \subseteq H$. Then $\bigcup \mathcal{S}$ is a filter over X such that $F \subseteq \bigcup \mathcal{S}$ for all $F \in \mathcal{S}$.*

How to use the lemma:

- At a limit $\lambda < \kappa$,

- let $G_\lambda = \bigcup\{G_\alpha \mid \alpha < \lambda\}$.

Once we have our full sequence $\langle G_\alpha \mid \alpha < \kappa\rangle$, let

$$G = \bigcup\{G_\alpha \mid \alpha < \kappa\}.$$

By Lemma 17, G is a filter extending F. Then G is an ultrafilter, because for all α, either A_α or $X - A_\alpha$ is in $G_{\alpha+1}$. ■

Proof of Lemma 17. We check each property of a filter.

- $\emptyset \notin \bigcup\mathcal{S}$: this is trivial. We have $A \in \bigcup\mathcal{S} \iff \exists F \in \mathcal{S}\ A \in F$, but $\emptyset \notin F$ for any $F \in \mathcal{S}$, so $\emptyset \notin \bigcup\mathcal{S}$.
- $X \in \bigcup\mathcal{S}$: there is $F \in \mathcal{S}$, a filter over X, and $X \in F$, so $X \in \bigcup\mathcal{S}$.
- For all $A \in \bigcup\mathcal{S}$ and $B \subseteq X$ such that $B \supseteq A$, we have $B \in \bigcup\mathcal{S}$.

  Let $A \in \bigcup\mathcal{S}$, $B \subseteq X$, $B \supseteq A$. Fix $F \in \mathcal{S}$ with $A \in F$. Then $B \in F$ since F is a filter, so $B \in \bigcup\mathcal{S}$.
- For all $A, B \in \bigcup\mathcal{S}$, $A \cap B \in \bigcup\mathcal{S}$.

  Let $F, G \in \mathcal{S}$ be such that $A \in F$ and $B \in G$. By the directedness hypothesis, take $H \in \mathcal{S}$ with $H \supseteq F, G$. So $A, B \in H$, and since H is a filter, $A \cap B \in H$. Hence $A \cap B \in \bigcup\mathcal{S}$. ■

2 The Club Filter

Definition. Let α be a limit ordinal.

- $A \subseteq \alpha$ is *unbounded in* α if $\forall\beta < \alpha\ \exists\gamma \in A\ \ \gamma \geq \beta$.
- $A \subseteq \alpha$ is *closed in* α if for all limit $\lambda < \alpha$ such that $A \cap \lambda$ is unbounded in λ, we have $\lambda \in A$.
- $A \subseteq \alpha$ is *club in* α if A is closed in α and unbounded in α.

Definition (Club filter on κ)**.** Let κ be regular. Let $F \subseteq \mathcal{P}(\kappa)$ be given by

$$A \in F \iff \text{there is } C \subseteq A \text{ such that } C \text{ is club in } \kappa,$$

i.e. A contains a club. F is called the *club filter on* κ.

Lemma 18. *The club filter F is a filter over κ, for $\kappa > \omega$ regular.*

Proof.

- $\emptyset \notin F$: the only subset $C \subseteq \emptyset$ is $C = \emptyset$, but $\emptyset$ is not club in κ.
- $\kappa \in F$: κ is club in κ.

- If $A \in F$ and $B \subseteq \kappa$ is such that $B \supseteq A$, then $B \in F$. Indeed, $A \in F$, so fix a club $C \subseteq A$. Then $B \supseteq A$ gives $C \subseteq B$, so $B \in F$.
- If $A, B \in F$, then $A \cap B \in F$. It is enough to show Lemma 19 below.

  Why? $A \in F$, so fix a club $C \subseteq A$; $B \in F$, so fix a club $D \subseteq B$. Then $C \cap D$ is club by the lemma, and $C \cap D \subseteq A \cap B$, so $A \cap B \in F$. ■

Lemma 19. *If C, D are club in a regular κ, then $C \cap D$ is club in κ.*

Proof. *$C \cap D$ is closed.* If $(C \cap D) \cap \lambda$ is unbounded in λ, then $C \cap \lambda$ and $D \cap \lambda$ are unbounded too. Since C and D are closed, $\lambda \in C$ and $\lambda \in D$, so $\lambda \in C \cap D$.

$C \cap D$ is unbounded. Fix $\xi < \kappa$. Build $\langle \alpha_i \mid i \in \omega \rangle$ in C and $\langle \beta_i \mid i \in \omega \rangle$ in D such that

$$\xi < \alpha_i < \beta_i < \alpha_{i+1}.$$

Note that, given such sequences, letting

$$\gamma = \sup\{\alpha_i \mid i \in \omega\} = \sup\{\beta_i \mid i \in \omega\},$$

$\{\alpha_i \mid i \in \omega\}$ witnesses that $C \cap \gamma$ is unbounded in γ, and $\{\beta_i \mid i \in \omega\}$ witnesses that $D \cap \gamma$ is unbounded in γ. By closure, $\gamma \in C$ and $\gamma \in D$, so we are done. Indeed $\xi < \gamma \in C \cap D$, and ξ was arbitrary.

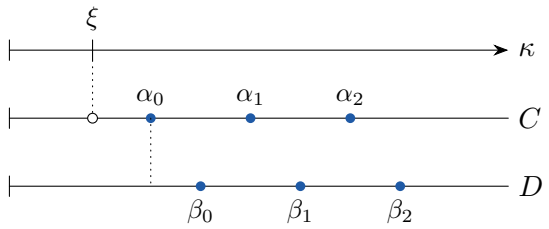

Figure 13: Interleaving the clubs C and D above ξ: $\xi < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$.

Because κ is regular and uncountable, these sequences still stay below κ. Let

$$\begin{aligned} \alpha_0 &= \min\big(C - (\xi + 1)\big), \\ \beta_i &= \min\big(D - (\alpha_i + 1)\big), \\ \alpha_{i+1} &= \min\big(C - (\beta_i + 1)\big). \end{aligned}$$

■

Lecture 14 — April 7, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 The Club Filter

Definition. The *club filter* on a regular cardinal $\kappa \geq \omega_1$, F_κ, is the set of all $X \subseteq \kappa$ such that there is $C \subseteq X$ with C club in κ (closed unbounded).

Fact 8 (last time)**.** The club filter is a filter on κ.

Definition. Let μ be an infinite cardinal. A filter F on X is *μ-complete* if and only if for all $S \subseteq F$ such that $|S| < \mu$, we have $\bigcap S \in F$.

Fact 9. F_κ is κ-complete.

(Exercise: it is not κ^+-complete.)

Fact 10. F_κ is not an ultrafilter.

Today we prove this for $\kappa > \omega_1$, which is easier. On Monday we will do $\kappa = \omega_1$, which is harder. (For ω_1 we need AC.)

There are sets X which contain a club but are not clubs themselves.

Note. For any club $C \subseteq \kappa$ and $\alpha < \kappa$, $C - \alpha$ is still club.

2 F_κ is κ-complete

Proof of Fact 9. Let $S \subseteq F_\kappa$ be such that $|S| < \kappa$. Let $\mu < \kappa$ and $\langle X_\alpha \mid \alpha < \mu \rangle$ be such that $S = \{X_\alpha \mid \alpha < \mu\}$. (These are chosen to enumerate S: let $|S| = \mu$, take $f : \mu \to S$ one-to-one and onto, and let $X_\alpha = f(\alpha)$.)

Note that $X_\alpha \in F_\kappa$, so there is some $C_\alpha \subseteq X_\alpha$ club in κ.

Observation. If we show that $\bigcap_{\alpha<\mu} C_\alpha$ is club, then $\bigcap_{\alpha<\mu} C_\alpha \subseteq \bigcap S$, so $\bigcap S \in F_\kappa$. This is the following lemma, with $\delta = \mu$. ■

Lemma 20. *Let κ be regular and uncountable, and let $\delta < \kappa$. For $\langle C_\alpha \mid \alpha < \delta \rangle$ a δ-sequence of club subsets of κ,*

$$\bigcap_{\alpha<\delta} C_\alpha \text{ is club.}$$

Proof. $\bigcap_{\alpha<\delta} C_\alpha$ is closed. Suppose $\lambda < \kappa$ is a limit and $(\bigcap_{\alpha<\delta} C_\alpha) \cap \lambda$ is unbounded in λ. We want to show $\lambda \in \bigcap_{\alpha<\delta} C_\alpha$. But for all $\alpha < \delta$,

$$\Big(\bigcap_{\alpha'<\delta} C_{\alpha'}\Big) \cap \lambda \subseteq C_\alpha \cap \lambda,$$

so $C_\alpha \cap \lambda$ is unbounded in λ. Since C_α is closed, $\lambda \in C_\alpha$. Hence $\lambda \in \bigcap_{\alpha<\delta} C_\alpha$.

$\bigcap_{\alpha<\delta} C_\alpha$ is unbounded in κ. We show

$$\forall \xi < \kappa \;\; \exists \gamma \in \bigcap_{\alpha<\delta} C_\alpha \;\; \gamma \geq \xi.$$

Idea. Fix $\xi < \kappa$. Build ω-sequences $\langle \beta_n^\alpha \mid n \in \omega \rangle$ from C_α, for $\alpha < \delta$, such that

$$\xi < \beta_n^\alpha < \beta_n^\eta < \beta_{n+1}^0 \qquad \text{for all } \alpha < \eta < \delta \text{ and } n \in \omega. \tag{$*$}$$

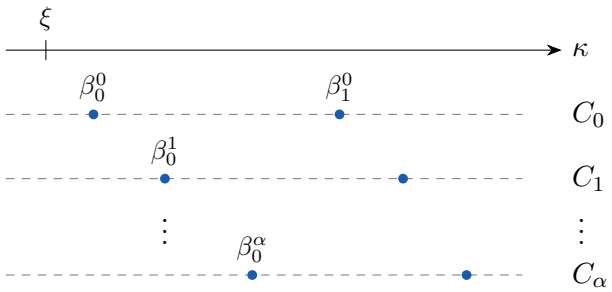

Figure 14: Choosing the β_n^α above ξ: within each round n they increase with α, and the next round starts above all of them.

Once we have done that, notice that for all $\alpha, \eta < \delta$,

$$\xi < \sup_{n\in\omega} \beta_n^\alpha = \sup_{n\in\omega} \beta_n^\eta.$$

Let $\gamma = \sup_{n\in\omega} \beta_n^\alpha$ for some $\alpha < \delta$ (by the above, this does not depend on α). Since κ is regular and uncountable, $\gamma < \kappa$. Then γ is a limit, and $\{\beta_n^\alpha \mid n \in \omega\} \subseteq C_\alpha \cap \gamma$ witnesses that $C_\alpha \cap \gamma$ is unbounded. So $\gamma \in C_\alpha$, as C_α is closed. This holds for every $\alpha < \delta$, so

$$\gamma \in \bigcap_{\alpha<\delta} C_\alpha,$$

and hence $\bigcap_{\alpha<\delta} C_\alpha$ is unbounded.

Construction. Let

$$\begin{aligned}
\beta_0^0 &= \min(C_0 - (\xi + 1)), \\
\beta_n^\alpha &= \min\Big(C_\alpha - \sup\{\beta_n^\eta + 1 \mid \eta < \alpha\}\Big) \quad \text{for } \alpha > 0, \\
\beta_{n+1}^0 &= \min\Big(C_0 - \sup\{\beta_n^\alpha + 1 \mid \alpha < \delta\}\Big).
\end{aligned}$$

Check that this gives that $(*)$ holds!

Claim 6. The β's are well-defined.

Since we are taking a minimum, we just need to check that each set is nonempty.

- β_0^0: $C_0 - (\xi + 1)$ is nonempty because C_0 is unbounded.

- β_n^α for $\alpha > 0$: we need $\sup\{\beta_n^\eta + 1 \mid \eta < \alpha\} < \kappa$. We have $\alpha < \delta < \kappa$, so $\{\beta_n^\eta + 1 \mid \eta < \alpha\}$ is bounded in κ because κ is regular. So since C_α is unbounded,

  $$C_\alpha - \sup\{\beta_n^\eta + 1 \mid \eta < \alpha\}$$

  is nonempty.

- Finally, for β_{n+1}^0 we just need $\{\beta_n^\alpha + 1 \mid \alpha < \delta\}$ to be bounded in κ. Since $\delta < \kappa$ and κ is regular, this is bounded. ■

3 F_κ is not an ultrafilter

Proof of Fact 10 for $\kappa = \omega_2$. We show that F_{ω_2} is not an ultrafilter (in fact the same works for any $\kappa \geq \omega_2$). This is on the homework.

- $C = \{\alpha < \omega_2 \mid \alpha \text{ is a limit ordinal}\}$ is club in ω_2.
- $S_0 = \{\alpha \in C \mid \mathrm{cf}(\alpha) = \omega\}$.
- $S_1 = \{\alpha \in C \mid \mathrm{cf}(\alpha) = \omega_1\}$.

On the homework you will show that $S_1 = C - S_0$ and that neither S_1 nor S_0 contains a club. Then neither S_0 nor $\omega_2 - S_0$ is in F_{ω_2}: if $\omega_2 - S_0$ contained a club D, then $D \cap C$ would be a club contained in S_1.

The trick: there are two distinct cofinalities below ω_2, namely ω_1 and ω. But this does not work for ω_1, since there is only one cofinality below it: ω.

21-329: Set Theory Spring 2023

Lecture 15 — April 10, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Recap

- The club filter F_κ, for κ a regular uncountable cardinal.
- F_κ is a κ-complete filter.
- On the homework: show it is not κ^+-complete.
- For $\kappa \geq \omega_2$, F_κ is not an ultrafilter. (Wednesday: F_{ω_1}, the club filter on ω_1, is not an ultrafilter.)

2 Normal Filters

Definition. A filter F on κ is *normal* if for any $\langle X_\alpha \mid \alpha < \kappa \rangle$ from F (i.e. $X_\alpha \in F$ for all $\alpha < \kappa$), the *diagonal intersection*

$$\triangle_{\alpha<\kappa} X_\alpha = \{\alpha < \kappa \mid \forall \beta < \alpha \;\; \alpha \in X_\beta\}$$

satisfies $\triangle_{\alpha<\kappa} X_\alpha \in F$. Here the condition on α says $\alpha \in \bigcap_{\beta<\alpha} X_\beta$, the intersection up to α.

Theorem 24. *For κ regular uncountable, the club filter F_κ on κ is normal.*

Lemma 21. *For $\langle C_\alpha \mid \alpha < \kappa \rangle$ club subsets of κ, $\triangle_{\alpha<\kappa} C_\alpha$ is club.*

Proof of Lemma 21. Fix $\langle C_\alpha \mid \alpha < \kappa \rangle$ club in κ. Let

$$D = \triangle_{\alpha<\kappa} C_\alpha = \{\alpha < \kappa \mid \forall \beta < \alpha \;\; \alpha \in C_\beta\}.$$

We'll show D is club.

Claim 7. D is closed.

Let $\lambda < \kappa$ be a limit ordinal such that $D \cap \lambda$ is unbounded in λ. We'll show $\lambda \in D$.

Suppose not, i.e. $\lambda \notin D$. By the definition of D, there is $\beta < \lambda$ with $\lambda \notin C_\beta$. So $C_\beta \cap \lambda$ is bounded in λ. (Otherwise, if $C_\beta \cap \lambda$ is unbounded in λ, then as C_β is closed, $\lambda \in C_\beta$, a contradiction.)

There is an $\alpha \in D \cap \lambda$ such that $\alpha > \beta$ and $\alpha > \sup(C_\beta \cap \lambda)$. Since $\alpha \in D$, we have $\alpha \in C_{\beta'}$ for all $\beta' < \alpha$, in particular $\alpha \in C_\beta$. Contradiction! (We have $\sup(C_\beta \cap \lambda) < \alpha < \lambda$, so in particular $\alpha \notin C_\beta$.)

Claim 8. D is unbounded.

(The usual approach: build an ω-sequence whose limit is in D.)

Fix $\xi < \kappa$. Build $\langle \beta_n \mid n \in \omega \rangle$ in κ, strictly increasing above ξ, so that for $\beta = \sup_{n\in\omega} \beta_n$, we have $\beta \in D$.

To get $\beta \in D$, we need $\beta \in C_\eta$ for all $\eta < \beta$. (If $\eta < \beta$, then $\eta < \beta_n$ for some n. We'll arrange the tail of the β_n's so that $\beta_{n+1} \in C_\eta$ whenever $\beta_n > \eta$. Then $\{\beta_{n+1} \mid \beta_n > \eta\}$ witnesses that $C_\eta \cap \beta$ is unbounded in β, so $\beta \in C_\eta$.)

Define β_n as follows, by recursion on ω:

$$\begin{aligned} \beta_0 &= \xi + 1, \\ \beta_{n+1} &= \min\Big(\bigcap_{\eta<\beta_n} C_\eta - (\beta_n + 1)\Big). \end{aligned}$$

Note. β_{n+1} is well-defined. Since $\beta_n < \kappa$, we have $|\{C_\eta \mid \eta < \beta_n\}| < \kappa$, so by κ-completeness

$$\bigcap\{C_\eta \mid \eta < \beta_n\} = \bigcap_{\eta<\beta_n} C_\eta$$

is club. In particular $\bigcap_{\eta<\beta_n} C_\eta$ is unbounded in κ, so $\min\big(\bigcap_{\eta<\beta_n} C_\eta - (\beta_n + 1)\big)$ is well-defined.

This construction works, since $\beta_n < \beta_{n+1}$ for all n, and clearly $\beta_{n+1} \in C_\eta$ for all $\eta < \beta_n$. Since κ is regular and uncountable, $\beta < \kappa$.

So D is club and we are done. ■

3 Stationary Sets and Fodor's Theorem

Definition. Let κ be an uncountable regular cardinal. A set $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club $C \subseteq \kappa$. (Equivalently, $S \cap X \neq \emptyset$ for all $X \in F_\kappa$.)

(Intuition: being stationary, that is, being compatible with clubs, is some sort of largeness property.)

Definition. For $S \subseteq \kappa$ and $f\colon S \to \kappa$, f is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in S$.

Theorem 25 (Fodor's Theorem)**.** *For κ regular uncountable, $S \subseteq \kappa$ stationary, and $f\colon S \to \kappa$ regressive, there is a stationary $T \subseteq S$ such that f is constant on T (i.e. there is $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$).*

Regressive functions are also called "pressing down" functions. Fodor's theorem says we can shrink down the stationary set so that f only outputs one value on it.

Proof. Fix S and f. Suppose f is not constant on any stationary $T \subseteq S$. This means that for any $\gamma < \kappa$,

$$f^{-1}[\{\gamma\}] = \{\alpha \in S \mid f(\alpha) = \gamma\} = T_\gamma$$

is not stationary, so there is C club in κ such that $C \cap T_\gamma = \emptyset$ (i.e. T_γ is disjoint from some club).

By AC, fix $\langle C_\gamma \mid \gamma < \kappa \rangle$ with C_γ club in κ and $C_\gamma \cap T_\gamma = \emptyset$. (So for all $\alpha \in C_\gamma$, $f(\alpha) \neq \gamma$, i.e. $\gamma \notin f[C_\gamma]$.)

Let

$$D = \bigtriangleup_{\gamma<\kappa} C_\gamma = \{\alpha < \kappa \mid \alpha \in C_\gamma \text{ for all } \gamma < \alpha\}.$$

D is club by Lemma 21. Since S is stationary, $S \cap D \neq \emptyset$. Fix $\alpha \in S \cap D$. (Recall $f\colon S \to \kappa$ with $f(\alpha) < \alpha$ for all $\alpha \in S$.)

- $\alpha \in S$, so $f(\alpha) < \alpha$.
- $\alpha \in D$, so $\alpha \in C_\gamma$ for all $\gamma < \alpha$.

In particular $\alpha \in C_{f(\alpha)}$. Contradiction: $f(\alpha) \notin f[C_{f(\alpha)}]$, but clearly $\alpha \in C_{f(\alpha)}$ implies $f(\alpha) \in f[C_{f(\alpha)}]$. ■

21-329: Set Theory Spring 2023

Lecture 16 — April 12, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Recall

For κ regular $> \omega$, the club filter F_κ on κ is *not* an ultrafilter.

For example, at ω_2 the sets

$$S = \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\},$$
$$T = \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\}$$

are *disjoint* stationary subsets of ω_2.

Proposition 5. Let κ be a regular uncountable cardinal. If S and T are disjoint stationary subsets of κ, then F_κ is not an ultrafilter.

Proof. Fix S, T. We'll show that neither S nor $\kappa - S$ can be in F_κ.

Suppose $S \in F_\kappa$. Then there is a club C with $C \subseteq S$. Since $T \cap S = \emptyset$, we get $C \cap T = \emptyset$. Contradiction! (T is stationary, so $T \cap C \neq \emptyset$.)

Suppose $\kappa - S \in F_\kappa$. Then there is a club C with $C \subseteq \kappa - S$, so $C \cap S = \emptyset$, contradicting that S is stationary. ■

2 Splitting ω_1

Theorem 26. *There is $S \subseteq \omega_1$ such that S and $\omega_1 - S$ are both stationary.*

Corollary 4. *F_{ω_1} is not an ultrafilter.*

(Both require AC.)

Proof of Theorem 26. Use AC to choose $\langle f_\alpha \mid \alpha < \omega_1 \rangle$, where $f_\alpha \colon \omega \to \alpha$ is a surjection.

Claim 9. For any $n < \omega$ there is $\theta < \omega_1$ such that

$$S^n_\theta = \{\alpha < \omega_1 \mid f_\alpha(n) = \theta\}$$

is stationary.

Proof. Fix n. Let $g(\alpha) = f_\alpha(n)$. Notice that g is regressive (as a function from ω_1 into ω_1). By Fodor's theorem, there is $\theta < \omega_1$ such that

$$\{\alpha \mid g(\alpha) = \theta\} = \{\alpha \mid f_\alpha(n) = \theta\}$$

is stationary. ■

Observation. If for some n there are distinct θ, γ such that S^n_θ and S^n_γ are stationary, then we're done.

(Why? $\theta \neq \gamma$ implies $S^n_\theta \cap S^n_\gamma = \emptyset$. So taking $S = S^n_\theta$, S is stationary, and $S^n_\gamma \subseteq \omega_1 - S$, so $\omega_1 - S$ is stationary too.)

Towards a contradiction, suppose that for every n there is a unique θ such that S^n_θ is stationary. Let $h\colon \omega \to \omega_1$ be given by

$$h(n) = \text{the unique } \theta \text{ such that } S^n_\theta \text{ is stationary.}$$

Claim 10. h is a surjection.

Proof. Suppose not. Fix $\theta < \omega_1$ such that $\theta \neq h(n)$ for any n. Then S^n_θ is *not* stationary for any n, i.e. $S^n_\theta \cap C = \emptyset$ for some club $C \subseteq \omega_1$.

Fix $\langle C_n \mid n \in \omega \rangle$ such that $C_n \subseteq \omega_1$ is club and $S^n_\theta \cap C_n = \emptyset$.

Recall the lemma: $\bigcap\{C_\alpha \mid \alpha < \lambda\}$ is club for any $\lambda < \kappa$ and clubs $C_\alpha \subseteq \kappa$. In particular, for $\kappa = \omega_1$,

$$D = \bigcap\{C_n \mid n \in \omega\}$$

is club.

For all $\alpha \in D$, $\alpha \notin S^n_\theta$ for any n, i.e. $f_\alpha(n) \neq \theta$ for any n.

D is unbounded, so there is $\alpha \in D$ with $\alpha > \theta$. But f_α is a surjection from ω onto α, so $\theta = f_\alpha(n)$ for some n. Contradiction! ■

But ω_1 is uncountable, so there is no surjection from ω onto ω_1. This contradiction proves the theorem. ■

Theorem 27 (Solovay's splitting theorem)**.** *Let κ be regular uncountable and $S \subseteq \kappa$ stationary. Then there is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ of pairwise disjoint stationary sets such that $S_\alpha \subseteq S$ for all $\alpha < \kappa$.*

(Idea: the club filter is very far from being an ultrafilter.)

3 Complete Ultrafilters

Question. Are there non-principal ultrafilters on a regular cardinal κ which are κ-complete? Normal? ω_1-complete (i.e. closed under countable intersections)?

Reference: Jech, *Set Theory*, chapter "Measurable cardinals".

Facts. If there is a countably complete non-principal ultrafilter on κ, then there is a γ-complete non-principal ultrafilter on γ for some $\gamma \leq \kappa$.

That is, the least κ such that there is a countably complete non-principal ultrafilter on κ also has a κ-complete non-principal ultrafilter on κ.

Theorem 28. *If there is a κ-complete non-principal ultrafilter on κ, then κ is* inaccessible*: a regular limit cardinal.*

Theorem 29 (Gödel)**.** *If there is a regular limit cardinal, then ZFC is consistent.*

(So if ZFC is consistent, ZFC can't show there is a regular limit cardinal.)

21-329: Set Theory Spring 2023

Lecture 17 — April 17, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Chapter 5: Trees

Sections 1–3 are essentially *descriptive set theory*: (very roughly) a set-theoretic study of the complexity of sets of real numbers, elaborating on classical topological notions.

2 Basics of Topology

Definition. A pair (S, T) is a *topological space* (or T is a *topology* on S) if

(1) $T \subseteq \mathcal{P}(S)$;

(2) $S \in T$;

(3) if $U, V \in T$, then $U \cap V \in T$;

(4) if $F \subseteq T$, then $\bigcup F \in T$.

If (S, T) is specified, we'll call the members of T the *open* subsets of S.

Example 8. Let $S = \mathbb{R}$, and let T be the topology generated by the open intervals (a, b) with $a < b$ reals. That is, $U \in T$ if and only if there is a set X of open intervals such that $U = \bigcup X$.

Proposition 6. $(\mathbb{R}, T)$ is a topological space.

Proof. We check the four conditions.

(1) Clear.

(2) $\mathbb{R} = \bigcup\{(a, b) \mid a < b\}$.

(3) If $U, V \in T$, then $U \cap V \in T$. Let

$$X = \{(a, b) \mid (a, b) \subseteq U\},$$
$$Y = \{(a, b) \mid (a, b) \subseteq V\}.$$

Observe that $\bigcup X = U$ and $\bigcup Y = V$.

(For instance, $\bigcup X \subseteq U$ by definition. To see $U \subseteq \bigcup X$, notice that $U = \bigcup Z$ for some set Z of open intervals. So for any $x \in U$, there is $(a, b) \in Z$ such that $x \in (a, b)$. But $(a, b) \subseteq U$, since $(a, b) \in Z$ and $\bigcup Z \subseteq U$, so $(a, b) \in X$, hence $x \in \bigcup X$.)

Claim 11. $U \cap V = \bigcup(X \cap Y)$.

Trivially $\bigcup(X \cap Y) \subseteq U \cap V$, so it suffices to show $U \cap V \subseteq \bigcup(X \cap Y)$.

Let $x \in U \cap V$. Then there are open intervals $(a, b) \subseteq U$, i.e. $(a, b) \in X$, such that $x \in (a, b)$, and $(c, d) \subseteq V$, i.e. $(c, d) \in Y$, such that $x \in (c, d)$.

By an easy case analysis, $(a, b) \cap (c, d)$ is an open interval containing x. But $(a, b) \cap (c, d) \subseteq (a, b) \subseteq U$, and similarly $(a, b) \cap (c, d) \subseteq (c, d) \subseteq V$, so

$$x \in (a, b) \cap (c, d) \in X \cap Y \implies x \in \bigcup(X \cap Y).$$

(4) Let $F \subseteq T$, and let

$$Y = \{(a, b) \mid (a, b) \subseteq U \text{ for some } U \in F\}.$$

Then $\bigcup F = \bigcup Y$, so $\bigcup F \in T$. ■

3 The Baire Space

Definition. The *Baire space* is $S = {}^{\omega}\omega$, with T the topology generated by the *basic open sets* N_s for

$$s \in {}^{<\omega}\omega = \bigcup_{n \in \omega} {}^{n}\omega,$$

where ${}^{n}\omega$ is the set of functions from n to ω. That is,

$$\begin{aligned} U \in T &\iff U = \bigcup X \text{ for some set } X \text{ of basic open sets} \\ &\iff U = \bigcup \{N_s \mid s \in I\} \text{ for some } I \subseteq {}^{<\omega}\omega, \end{aligned}$$

where

$$N_s = \{x \in {}^{\omega}\omega \mid x \supseteq s\} = \{x \in {}^{\omega}\omega \mid x \restriction |s| = s\},$$

the set of x that have s as their initial segment of length $|s|$.

3.1 Tree perspective

${}^{\omega}\omega$ is the set of infinite paths through the "tree" ${}^{<\omega}\omega$, and N_s is the set of infinite paths in ${}^{<\omega}\omega$ which extend s.

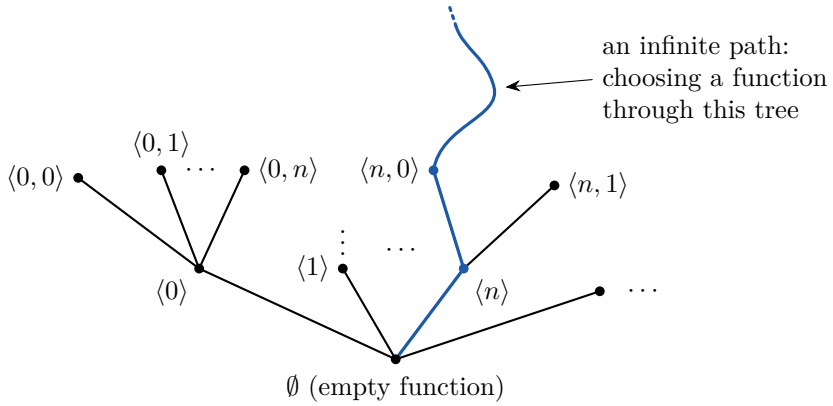

Figure 15: The tree ${}^{<\omega}\omega$ of finite sequences, with an infinite path through it.

A sequence of length $n+1$ is the same thing as a function from $n+1$ into ω.

Definition. s is a *predecessor* of t if and only if $s \subseteq t$.

${}^{\omega}\omega$ is (literally) the set of infinite paths through ${}^{<\omega}\omega$, and N_s is the set of branches through the subtree

$$\{t \in {}^{<\omega}\omega \mid t \subseteq s \text{ or } s \subseteq t\}.$$

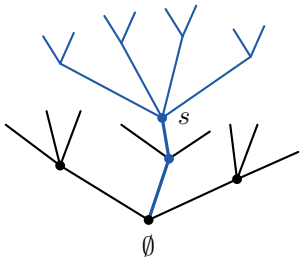

Figure 16: N_s consists of the branches through the subtree of nodes comparable with s (highlighted).

Proposition 7. $({}^{\omega}\omega, T)$ is a topological space.

Proof. The only nontrivial clause is (3): if $U, V \in T$, then $U \cap V \in T$. Let

$$I = \{s \in {}^{<\omega}\omega \mid N_s \subseteq U\},$$
$$J = \{t \in {}^{<\omega}\omega \mid N_t \subseteq V\}.$$

Check:

$$\bigcup_{s \in I} N_s = U, \qquad \bigcup_{t \in J} N_t = V.$$

Claim 12. $U \cap V = \bigcup_{s \in I \cap J} N_s$.

It is trivial to get $\bigcup_{s \in I \cap J} N_s \subseteq U \cap V$, so we just need to see

$$U \cap V \subseteq \bigcup_{s \in I \cap J} N_s.$$

Let $x \in U \cap V$. Then $x \in N_s$ for some $s \in I$ and $x \in N_t$ for some $t \in J$. So $s = x \restriction |s|$ and $t = x \restriction |t|$, and either $s \subseteq t$ or $t \subseteq s$. Without loss of generality, $t \subseteq s$.

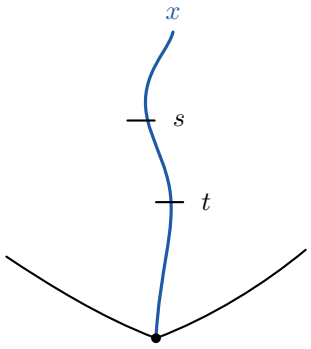

Figure 17: The branch x passes through both t and s, with $t \subseteq s$.

It is easy to check that $N_s \subseteq N_t$. Then

$$s \in I,\ t \in J \implies N_t \subseteq V \implies N_s \subseteq V \implies s \in J.$$

So $s \in I \cap J$ and $x \in N_s$, hence $x \in \bigcup_{s \in I \cap J} N_s$. ■

21-329: Set Theory Spring 2023

Lecture 18 — April 19, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

Announcement. Homework 9 (on Chapter 5) is due next Friday.

1 Last Time

- The definition of topological spaces, and examples.
- The *Baire space* ${}^{\omega}\omega$, with the topology generated by the basic open sets

$$N_s = \{x \in {}^{\omega}\omega \mid x \supseteq s\} \qquad \text{for } s \in {}^{<\omega}\omega.$$

2 Metric Spaces

Definition. A *metric space* is a pair (S, d), where $d\colon S \times S \to [0, \infty)$ satisfies, for all $x, y, z \in S$,

(1) $d(x, y) = d(y, x)$;

(2) $d(x, y) = 0 \iff x = y$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Example 9. $S = \mathbb{R}$ with $d(x, y) = |x - y|$.

Given a metric space (S, d), we can form a topological space (S, T) by taking as basic open sets the *open balls* (relative to d):

$$B_r(c) = \{x \in S \mid d(x, c) < r\}, \qquad r \in (0, \infty) \text{ and } c \in S.$$

That is,

$$T = \left\{U \subseteq S \;\middle|\; \text{there is a set } X \text{ of pairs } (c, r) \in S \times (0, \infty) \text{ such that } U = \bigcup_{(c,r)\in X} B_r(c)\right\}.$$

Note. For $(\mathbb{R}, d)$ as in Example 9, the resulting topology on $\mathbb{R}$ is the usual one, i.e. the one generated by open intervals.

Why? The balls $B_r(c)$ are exactly the open intervals. Given an interval (a, b) with $a < b$, let $c = \frac{a+b}{2}$ and $r = \frac{b-a}{2}$; then $(a, b) = B_r(c)$.

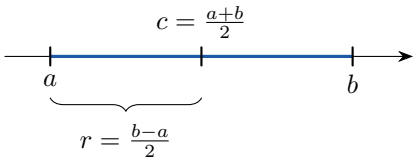

Figure 18: The interval (a, b) is the ball of radius r around its midpoint c.

And given r, c, we have $B_r(c) = (c - r, c + r)$.

2.1 The Baire space as a metric space

Define $d\colon {}^{\omega}\omega \times {}^{\omega}\omega \to [0, \infty)$ by: for $x, y \in {}^{\omega}\omega$,

$$d(x,y) = \begin{cases} \dfrac{1}{2^n} & \text{if } x \neq y, \text{ for } n \text{ least such that } x(n) \neq y(n), \\ 0 & \text{if } x = y. \end{cases}$$

If x and y are different, they agree up to some point and then diverge: $x \restriction n = y \restriction n$ but $x(n) \neq y(n)$.

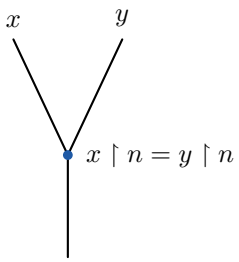

Figure 19: Two different branches x and y agree up to n and then diverge.

On the homework:

- Verify that $({}^{\omega}\omega, d)$ is a metric space.
- The induced topological space is the Baire space. For this, it's enough to see that the basic open sets N_s for $s \in {}^{<\omega}\omega$ are exactly the open balls $B_r(c)$ for $r \in (0, \infty)$ and $c \in {}^{\omega}\omega$.

3 Closed Sets and Trees

Definition. For (S, T) a topological space, $U \subseteq S$ is *open* (with respect to T) if $U \in T$, and $C \subseteq S$ is *closed* (with respect to T) if $S - C$ is open, i.e. $S - C \in T$.

Theorem 30. *A set $C \subseteq {}^{\omega}\omega$ is closed if and only if there is a tree T on ω such that C is the set of all infinite paths (branches) through T (i.e. $C = [T]$).*

Definition. A *tree* on a set Ω is a subset $T \subseteq {}^{<\omega}\Omega$ which is downwards closed, i.e. if $s \in T$ and $n < |s|$, then $s \restriction n \in T$.

We'll only care about trees on ω.

Example 10 (Trees)**.** The following are all trees:

- ${}^{<\omega}\omega$, with $[{}^{<\omega}\omega] = {}^{\omega}\omega$;
- $T_s = \{t \in {}^{<\omega}\omega \mid t \supseteq s \text{ or } t \subseteq s\}$ for any $s \in {}^{<\omega}\omega$, with $[T_s] = N_s$;
- $T_x = \{t \in {}^{<\omega}\omega \mid t \subseteq x\}$ for any $x \in {}^{\omega}\omega$, with $[T_x] = \{x\}$.

Definition. For T a tree on ω, $[T]$ is the set of all infinite paths (branches) through T:

$$[T] = \{x \in {}^{\omega}\omega \mid \forall n \in \omega \;\; x \restriction n \in T\}.$$

(We really only care about infinite trees T.)

Notice. For any $s \in {}^{<\omega}\omega$, N_s is closed. (Since it is also open, we call it *clopen*.) Why?

$${}^{\omega}\omega - N_s = \bigcup\{N_t \mid t \not\supseteq s, \;\; s \not\supseteq t\}.$$

Proof of Theorem 30. ($\Leftarrow$) Let T be a tree on ω. We'll show $[T]$ is closed. To do this, we show ${}^{\omega}\omega - [T]$ is open:

$${}^{\omega}\omega - [T] = \bigcup\{N_s \mid s \notin T\}.$$

If $x \in {}^{\omega}\omega - [T]$, then $x \restriction n \notin T$ for some $n \in \omega$. But then $x \in N_{x\restriction n}$, so x is in the right-hand side.

If $s \notin T$, then $N_s \cap [T] = \emptyset$ (so $N_s \subseteq {}^{\omega}\omega - [T]$), because for any $x \in N_s$,

$$s = x \restriction n \text{ for some } n \implies x \restriction n \notin T \implies x \notin [T].$$

($\Rightarrow$) Let $C \subseteq {}^{\omega}\omega$ be closed. Let

$$T = \{s \in {}^{<\omega}\omega \mid s \subseteq x \text{ for some } x \in C\} = \{x \restriction n \mid x \in C, \; n \in \omega\}.$$

Check: T is a tree on ω.

Claim 13. $[T] = C$.

$C \subseteq [T]$ trivially (for all $x \in C$, $x \restriction n \in T$ for all $n \in \omega$). So we just need to show $[T] \subseteq C$.

Suppose not: we have $x \in [T] - C$. Let $U = {}^{\omega}\omega - C$, so U is open. Then $x \in [T] - C$ implies $x \in U$. But then we can take $s \in {}^{<\omega}\omega$ such that $x \in N_s$ and $N_s \subseteq U$.

Letting $n = |s|$, we have $s = x \restriction n \in T$. But then by the definition of T, there is $y \in C$ such that $y \restriction n = s = x \restriction n$. So

$$y \in N_s \subseteq U = {}^{\omega}\omega - C,$$

but $y \in C$, a contradiction. ■

21-329: Set Theory Spring 2023

Lecture 19 — April 21, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

Announcements.

- Homework 9 is the last official homework. It is due next Friday, 4/28.
- Extra credit Homework 10 is to be announced, on next week's material.
- There will also be an extra credit problem on the final exam on next week's material.

1 Last Time

A set $C \subseteq {}^{\omega}\omega$ is closed if and only if there is a tree T on ω such that $C = [T]$.

Theorem 31 (Cantor)**.** *For $C \subseteq {}^{\omega}\omega$ closed, either C is countable or $|C| = 2^{\aleph_0}$.*

Cantor proved this as part of a plan to show

$$\underbrace{2^{\aleph_0} = \omega_1}_{\text{“CH”}},$$

which we now know is "impossible".

2 Perfect Trees and Perfect Sets

Definition. A tree T on ω is *perfect* if for all $s \in T$ there are $u, v \in T$ such that

- $u \perp v$ (i.e. $u \not\supseteq v$ and $u \not\subseteq v$), and
- $u, v \supseteq s$.

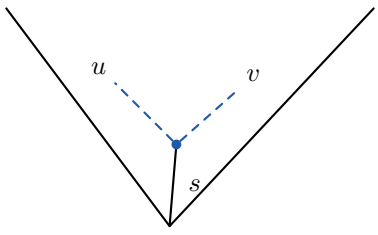

Figure 20: In a perfect tree, every node s has two incompatible extensions u and v.

Definition. A set $P \subseteq {}^{\omega}\omega$ is *perfect* if $P = [T]$ for T a perfect tree.

Example 11. ${}^{\omega}\omega$ is perfect, because $T = {}^{<\omega}\omega$ is perfect and $[T] = {}^{\omega}\omega$.

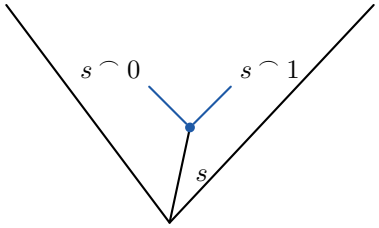

Figure 21: In ${}^{<\omega}\omega$, every node s has the incompatible extensions $s \frown 0$ and $s \frown 1$.

Example 12. ${}^{\omega}2$ is perfect, because $T = {}^{<\omega}2$ is perfect and $[{}^{<\omega}2] = {}^{\omega}2$.

Non-example. For any $x \in {}^{\omega}\omega$, $\{x\}$ is closed but not perfect.

Recall $\{x\} = [T_x]$, where T_x is the tree of initial segments of x. T_x is not perfect: $\emptyset \in T_x$, but any extensions of $\emptyset$ in T_x are comparable (because they are initial segments of x).

In fact, for any T such that $\{x\} = [T]$, you can show T is not perfect. So $\{x\}$ is not perfect.

On the homework you'll show: if P is perfect and nonempty, then $|P| = 2^{\aleph_0}$.

Hint. Let P be perfect and nonempty. Fix a perfect tree T such that $P = [T]$. You should produce a one-to-one map $\pi\colon {}^{<\omega}2 \to T$ so that, letting

$$\pi^*(x) = \bigcup_{s \subseteq x} \pi(s),$$

the map $\pi^*\colon {}^{\omega}2 \to [T]$ is one-to-one.

3 The Derivative of a Tree

Definition. For T a tree on ω, the *derivative* of T is

$$T' = \{s \in T \mid N_s \cap [T] \text{ has at least two elements}\}.$$

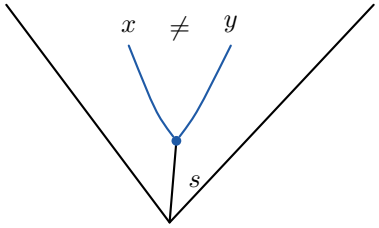

Figure 22: A node s is in T' when at least two different branches $x \neq y$ of T pass through s.

Observation. If T is a tree on ω, so is T', and $T' \subseteq T$, so $[T'] \subseteq [T]$.

4 Closed Sets Are Countable or Contain a Perfect Set

Theorem 32. *For $C \subseteq {}^{\omega}\omega$ closed, either C is countable or there is a nonempty perfect $P \subseteq C$.*

Proof. Fix T^0 such that $[T^0] = C$. We'll define trees T^α on ω for any ordinal α. Let

- $T^{\alpha+1} = (T^\alpha)'$, the derivative of T^α;
- $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for λ a limit.

Observation. If $\alpha < \beta$, then $T^\beta \subseteq T^\alpha$, so $[T^\beta] \subseteq [T^\alpha]$.

Claim 14. For some countable ordinal δ (i.e. $\delta < \omega_1$), $T^{\delta+1} = T^\delta$. (So $T^\beta = T^\delta$ for all $\beta \geq \delta$.)

Proof of Claim 14. Suppose not. Then for all $\alpha < \omega_1$, $T^{\alpha+1} \neq T^\alpha$, i.e. $(T^\alpha)' \neq T^\alpha$. This means there is an $s \in T^\alpha$ such that $s \notin (T^\alpha)'$. So either

$$N_s \cap [T^\alpha] = \emptyset \qquad \text{or} \qquad \exists!\, x \in N_s \cap [T^\alpha].$$

Fix s_α some such s.

Notice that if $\alpha < \beta$, then $s_\alpha \neq s_\beta$, because $s_\alpha \notin T^{\alpha+1} \supseteq T^\beta$ while $s_\beta \in T^\beta$.

But now the map $\alpha \mapsto s_\alpha$ is a one-to-one map from ω_1 into ${}^{<\omega}\omega$. But ${}^{<\omega}\omega$ is countable, a contradiction. ■

Back to the proof. Fix δ as in Claim 14.

Claim 15. T^δ is a perfect tree.

Proof of Claim 15. Let $s \in T^\delta$. We need to show there are $u, v \in T^\delta$ such that $u, v \supseteq s$ and $u \perp v$.

Look at $N_s \cap [T^\delta]$. That $(T^\delta)' = T^\delta$ tells us $N_s \cap [T^\delta]$ has (at least) two distinct elements x, y. Let n be least such that $x(n) \neq y(n)$. Then

- $x \restriction (n+1),\ y \restriction (n+1) \in T^\delta$;
- $x \restriction (n+1),\ y \restriction (n+1) \supseteq s$ (they cannot diverge below s);
- but $x \restriction (n+1) \perp y \restriction (n+1)$.

So $u = x \restriction (n+1)$ and $v = y \restriction (n+1)$ work. Hence T^δ is a perfect tree. ■

If $[T^\delta]$ is nonempty, we're done, as $[T^\delta]$ is a nonempty perfect subset of C (because $T^\delta \subseteq T^0$ and $[T^0] = C$).

Claim 16. If $[T^\delta] = \emptyset$, then C is countable.

Proof of Claim 16. For any $x \in C$ there is an $\alpha < \delta$ such that $x \in [T^\alpha]$ but $x \notin [T^{\alpha+1}]$, and this α is unique; call it α_x.

Since $T^{\alpha+1} = (T^\alpha)'$, we have $x \notin [(T^\alpha)']$, so for some $s = x \restriction n$ with $n \in \omega$, $s \notin (T^\alpha)'$. But then $N_s \cap [T^\alpha]$ has at most one element. But then actually $N_s \cap [T^\alpha] = \{x\}$. Let s_x be the shortest such s.

Suppose $x, y \in C$ with $x \neq y$. If $\alpha_x = \alpha_y$, then $s_x \neq s_y$, as x is the unique element of $N_{s_x} \cap [T^{\alpha_x}]$.

So the map $f(x) = (\alpha_x, s_x)$, for $x \in C$, is one-to-one. Moreover

$$\text{range}(f) \subseteq \delta \times {}^{<\omega}\omega,$$

where δ is countable and ${}^{<\omega}\omega$ is countable, so $\delta \times {}^{<\omega}\omega$ is countable. Hence the range of f is countable, and since f is one-to-one, C is countable. ■

Overall idea: transfinite iteration of the derivative operator.

21-329: Set Theory Spring 2023

Lecture 20 — April 24, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

Announcements.

- The final is cumulative; there are review problems.
- Homework 9 is due Friday.

1 König's Theorem

Theorem 33 (König)**.** *Let T be a tree on ω such that T is infinite and, for all $n \in \omega$,*

$$\{s \in T \mid s \in {}^{n}\omega\}$$

is finite. Then T has an infinite path, i.e. $[T] \neq \emptyset$.

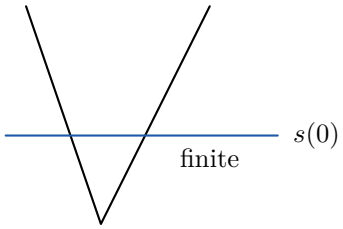

Figure 23: Each level of T, such as the level of the values $s(0)$, is finite.

Corollary 5. *Let T be a tree on ω such that $T \subseteq {}^{<\omega}2$ and T is infinite. Then T has an infinite path.*

Definition. For T a tree on ω and $s \in {}^{<\omega}\omega$,

$$T_s = \{t \in T \mid t \subseteq s \text{ or } t \supseteq s\},$$

the set of all elements of T that are comparable to s.

Definition (Successor)**.** t is an *immediate successor* of s if $|t| = |s| + 1$ and $s \subseteq t$.

Lemma 22. *Let T be a tree on ω and $s \in T$ such that T_s is infinite and*

$$\{t \in T \mid t \text{ is an immediate successor of } s\}$$

is finite. Then there is an immediate successor t of s such that $t \in T$ and T_t is infinite.

Proof. Let $s \in T$.

Observation.

$$T_s = \bigcup \{T_t \mid t \in T \text{ an immediate successor of } s\}.$$

Proof of the observation. ($\supseteq$) is clear.

($\subseteq$) Let $r \in T_s$. If $r \subseteq s$, then $r \subseteq t$ for any immediate successor t of s, so $r \in T_t$. (Such a $t \in T$ exists here, since T_s is infinite, so s has a proper extension in T.)

If $s \subseteq r$ but $r \not\subseteq s$, then $|r| \geq |s| + 1$ (i.e. there is $n \in \mathrm{dom}(r)$ with $r \restriction n = s$). But then $r \restriction (|s| + 1)$ is an immediate successor of s in T; call it t. So $r \in T_t$.

Now suppose the lemma fails, i.e. for all immediate successors $t \in T$ of s, T_t is finite. By the observation and the hypothesis, T_s is a finite union of finite sets, so T_s is finite, a contradiction. ■

Proof of König's Theorem. We'll build a sequence $\langle s_n \mid n \in \omega \rangle$ by recursion such that

- $s_n \in T$ for all n;
- s_{n+1} is an immediate successor of s_n;
- T_{s_n} is infinite;
- $|s_n| = n$.

Base step. $s_0 = \emptyset$. ($T_\emptyset = T$ is infinite.)

Induction step. Given s_n, we need to define s_{n+1}. T_{s_n} is infinite, and

$$\{t \in T \mid t \text{ is an immediate successor of } s_n\}$$

is finite, because it is a subset of $\{t \in T \mid t \in {}^{|s_n|+1}\omega\}$, which is finite by hypothesis. Let s_{n+1} be any $t \in T$, t an immediate successor of s_n, such that T_t is infinite (one exists by Lemma 22).

Let $x = \bigcup_n s_n \colon \omega \to \omega$. Then $x \restriction n = s_n \in T$ for all n, so $x \in [T]$. ■

Example 13 (Informal)**.** Let

$$T = \{s \in {}^{<\omega}\omega \mid \exists n \in \omega \;\; |s| \leq n \text{ and } \forall i < |s| \;\; s(i) = n\},$$

so all nodes are constant. Show that T is infinite but has no infinite paths.

2 Wellfounded Trees and Rank Functions

Definition. A tree T on ω is *wellfounded* if $[T] = \emptyset$ (T has no infinite paths). T is *illfounded* if $[T] \neq \emptyset$.

Definition. A *rank function* on a tree T is a function

$$\rho \colon T \to \mathrm{Ord}$$

(with domain T, such that $\rho(s)$ is an ordinal for all $s \in T$) such that for all $s, t \in T$,

$$s \subsetneq t \implies \rho(s) > \rho(t)$$

(here $s \subsetneq t$ means $s \subseteq t$ but $t \not\subseteq s$).

(The values of ρ are bounded in the ordinals: there is no unbounded set of ordinals.)

Theorem 34. *T is wellfounded if and only if there is a rank function ρ on T such that $\rho\colon T \to \omega_1$.*

Proposition 8 (Easy direction)**.** If T has a rank function, then T is wellfounded.

Proof. Let ρ be a rank function on T. Suppose towards a contradiction that T is illfounded, i.e. $[T] \neq \emptyset$. Let $x \in [T]$, so $x \restriction n \in T$ for all $n \in \omega$. Look at $\{\rho(x \restriction n) \mid n \in \omega\}$. Since

$$x \restriction (n+1) \supsetneq x \restriction n \implies \rho(x \restriction (n+1)) < \rho(x \restriction n),$$

the set

$$S = \{\rho(x \restriction n) \mid n \in \omega\}$$

doesn't have a least element: if $\alpha = \rho(x \restriction n)$, then $\rho(x \restriction (n+1)) < \alpha$. But S is a set of ordinals, a contradiction! ■

21-329: Set Theory Spring 2023

Lecture 21 — April 26, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Recall

- A tree T on ω is *wellfounded* if $[T] = \emptyset$.
- ρ is a *rank function* on T if $\mathrm{dom}(\rho) = T$, for all $s \in T$, $\rho(s)$ is an ordinal, and for all $s, t \in T$ such that $s \subsetneq t$, $\rho(s) > \rho(t)$.

Theorem 35. *T is wellfounded if and only if it has a rank function.*

2 Products of the Baire Space

${}^{\omega}\omega \times {}^{\omega}\omega$ has a natural topology (the *product topology*) coming from the topology on ${}^{\omega}\omega$. Its basic open sets are N_s for $s \in {}^{<\omega}(\omega \times \omega)$ (finite sequences of pairs of natural numbers).

For T a tree on $\omega \times \omega$, let

$$[T] = \{x \in {}^{\omega}(\omega \times \omega) \mid \forall n \;\; x \restriction n \in T\}.$$

Recall. In the Baire space, $C \subseteq {}^{\omega}\omega$ is closed if and only if $C = [T]$ for some tree T on ω.

In the product space, C is closed if and only if $C = [T]$ for some tree T on $\omega \times \omega$.

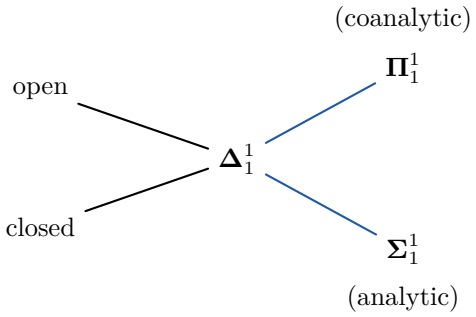

Figure 24: Open and closed sets are $\mathbf{\Delta}^1_1$, and $\mathbf{\Delta}^1_1$ sets are both $\mathbf{\Sigma}^1_1$ (analytic) and $\mathbf{\Pi}^1_1$ (coanalytic).

Definition. $A \subseteq {}^{\omega}\omega$ is (boldface) $\mathbf{\Sigma}^1_1$ if and only if there is a closed set $C \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ such that

$$A = \{x \in {}^{\omega}\omega \mid \exists y \in {}^{\omega}\omega \;\; (x, y) \in C\}.$$

(A is the projection of C onto the first coordinate.)

$B \subseteq {}^{\omega}\omega$ is $\mathbf{\Pi}^1_1$ if and only if ${}^{\omega}\omega - B$ is $\mathbf{\Sigma}^1_1$.

For T a tree on $\Omega \times \Gamma$, where Ω and Γ are two arbitrary sets, let

- $[T] = \{x \in {}^{\omega}(\Omega \times \Gamma) \mid x \restriction n \in T \text{ for all } n\}$;
- $p[T] = \{x \in {}^{\omega}\Omega \mid \exists f \in {}^{\omega}\Gamma \ \ (x, f) \in [T]\}$.

Here x is the first coordinate and f is the second coordinate.

Notice. A is $\mathbf{\Sigma}^1_1$ if and only if $A = p[T]$ for some tree T on $\omega \times \omega$.

Fact 11. Open sets and closed sets are both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. The notation for being both is $\mathbf{\Delta}^1_1$.

The $\mathbf{\Delta}^1_1$ subsets of the Baire space are the *Borel* subsets of the Baire space.

The theorem on wellfounded trees (that T is wellfounded if and only if there is a rank function ρ on T such that $\rho\colon T \to \omega_1$) gives:

If $B \subseteq {}^{\omega}\omega$ is $\mathbf{\Pi}^1_1$, then there is a tree T on $\omega \times \omega_1$ such that $B = p[T]$.

(This is why the rank function is important.)

3 Determinacy (§5.4)

Given $A \subseteq {}^{\omega}\omega$, we associate a two-player game G_A. There are two players, I and II, who alternate moves, playing elements of ω:

I	$x(0)$		$x(2)$		$\cdots$
II		$x(1)$		$x(3)$	$\cdots$

with $x(i) \in \omega$ for all i (the players alternate picking natural numbers). So $x \in {}^{\omega}\omega$.

In a round of G_A, I and II build an $x \in {}^{\omega}\omega$. I wins the round if $x \in A$ (II wins if $x \notin A$).

Example 14. $A = \{x \in {}^{\omega}\omega \mid x \text{ is surjective}\}$. I wins a round if x is surjective.

Observation. I can always win G_A. Why? I could just always play $x(2n) = n$.

Example 15 (A game where II wins)**.** $A = \{x \in {}^{\omega}\omega \mid x \text{ is injective}\}$. II wins if x is *not* injective.

Observation. II has a winning strategy in G_A:

- II could just always let $x(2n + 1) = 0$.
- II could also just copy I's moves: let $x(2n + 1) = x(2n)$.

Definition. A (or G_A) is *determined* if either I or II has a winning strategy in G_A.

Theorem 36 (Gale–Stewart)**.** *If $A \subseteq {}^{\omega}\omega$ is closed, then A is determined.*

Theorem 37 (Gale–Stewart; needs AC)**.** *There is $A \subseteq {}^{\omega}\omega$ such that A is* not *determined.*

(Idea: closed sets are simple, so they are determined. But sets are not determined in general.)

Theorem 38 (Martin)**.** *If A is $\mathbf{\Delta}^1_1$ (Borel), then A is determined.*

Theorem 39 (Martin)**.** *Assume there is a countably complete non-principal ultrafilter on some regular $\kappa > \omega$. Then for any $A \subseteq {}^\omega\omega$ that is $\mathbf{\Pi}^1_1$ (or $\mathbf{\Sigma}^1_1$), A is determined.*

(This is independent of ZFC.)

(Idea: a very uncountable large ordinal affects the determinacy of something close to the reals.)

21-329: Set Theory Spring 2023

Lecture 22 — April 28, 2023

Dr. Benjamin Siskind *Scribe: Fan Pu Zeng*

1 Closed Games

Theorem 40 (Gale–Stewart)**.** *Closed games are determined. (For any closed $A \subseteq {}^{\omega}\omega$, either* I *or* II *has a winning strategy in G_A.)*

Recall the game G_A:

$$G_A: \quad \begin{array}{c|ccccc} \text{I} & x(0) & & x(2) & \cdots \\ \hline \text{II} & & x(1) & & \cdots \end{array}$$

I wins if and only if $x \in A$.

2 Strategies

Definition. A *strategy* is a function $\sigma : {}^{<\omega}\omega \to \omega$.

For $b \in {}^{\omega}\omega$, $\sigma * b$ is the unique element of ${}^{\omega}\omega$ such that

- $\sigma * b\,(2k+1) = b(k)$;
- $\sigma * b\,(2k) = \sigma(\sigma * b \restriction 2k)$.

(Player II plays the entries of b.)

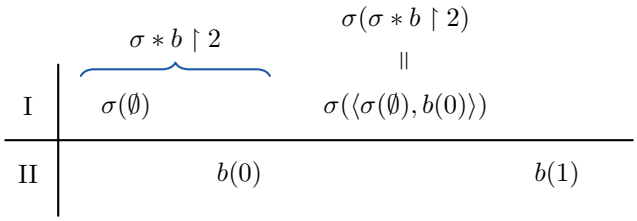

Figure 25: The play $\sigma * b$: player I follows σ and player II plays $b(0), b(1), \ldots$.

Definition (Winning strategy)**.** σ is a *winning strategy for* I in G_A if and only if for any $b \in {}^{\omega}\omega$, $\sigma * b \in A$. (Whenever player I plays according to σ, player I wins G_A.)

For $a \in {}^{\omega}\omega$, $a * \sigma$ is the unique element of ${}^{\omega}\omega$ such that

- $a * \sigma\,(2k) = a(k)$;

- $a * \sigma\,(2k+1) = \sigma(a * \sigma \restriction (2k+1))$.

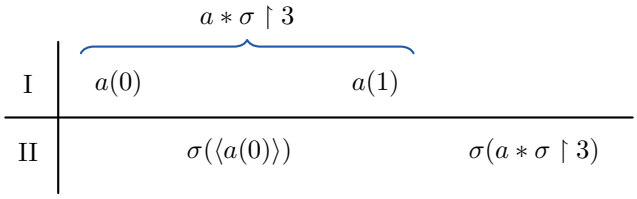

Figure 26: The play $a * \sigma$: player I plays $a(0), a(1), \ldots$ and player II follows σ.

Definition. σ is a *winning strategy for* II in G_A if and only if for any $a \in {}^{\omega}\omega$, $a * \sigma \notin A$. (Whenever II plays according to σ, II wins G_A.)

3 The Gale–Stewart Theorem

Theorem 41 (Gale–Stewart)**.** *For $A \subseteq {}^{\omega}\omega$ closed, either* I *or* II *has a winning strategy in G_A.*

Proof. Let $A \subseteq {}^{\omega}\omega$ be closed. Let T be a tree on ω such that $A = [T]$.

Suppose player II does not have a winning strategy in G_A. We'll describe a winning strategy for I.

Call $s \in {}^{<\omega}\omega$ a *winning position for* II if $|s|$ is even and II has a winning strategy in the game $G_{[T_s]}$. (Recall: $T_s = \{t \in T \mid t \supseteq s \text{ or } s \supseteq t\}$, so $T_s \subseteq T$ and $[T_s] \subseteq [T]$.)

Examples of winning positions for II.

- If $s \notin T$, then s is winning for II.
- If $[T_s] = \emptyset$, then s is winning for II.

Observation 1. $\emptyset$ is *not* a winning position for II, because II doesn't have a winning strategy in $G_A = G_{[T_\emptyset]}$, by assumption.

Observation 2. For any $s \in {}^{<\omega}\omega$ of even length which is not winning for II, there is $k \in \omega$ such that for any $l \in \omega$, $s \frown \langle k, l\rangle$ is not winning for II.

Proof of Observation 2. Suppose not, i.e. for any k there is an l such that $s \frown \langle k, l\rangle$ is a winning position for II. We'll describe a winning strategy for II in $G_{[T_s]}$, contradicting that s is not a winning position.

For any k, fix $l_k \in \omega$ and $\sigma_k \colon {}^{<\omega}\omega \to \omega$ such that σ_k is a winning strategy for II in $G_{[T_{s \frown \langle k, l_k\rangle}]}$. But then the following strategy is winning for II in $G_{[T_s]}$: in the round where I plays k (right after the position s), II responds with l_k and then plays according to σ_k. Since σ_k is winning in $G_{[T_{s \frown \langle k, l_k\rangle}]}$ for all $k \in \omega$, this is a winning strategy for II in $G_{[T_s]}$. Contradiction.

Observation 3. If s (of even length) is not winning for II, then $s \in T$.

Now the following is a winning strategy for I. For s of even length:

$$\sigma(s) = \begin{cases} \text{least } k \text{ such that } s \frown \langle k, l\rangle \text{ is not winning for any } l \in \omega, & \text{if } s \text{ is not winning for II,} \\ 0 & \text{if } s \text{ is winning for II.} \end{cases}$$

Why is this a winning strategy for I? Show by induction that for any $b \in {}^{\omega}\omega$, $\sigma * b \restriction 2k$ is *not* winning for II for all $k \in \omega$. By Observations 1–3,

$$\sigma * b \restriction 2k \in T \text{ for all } k \implies \sigma * b \in [T] = A.$$

So σ is winning for I. ■

(This is the beginning of descriptive set theory: determinacy.)